Performing these operations on the determinant A we get

$$\left[\frac{\partial A}{\partial R}\right]_{R_2=0} \equiv \left[\frac{\partial A}{\partial R_2}\right]_{R=0}$$

and

$$\left.\begin{array}{r} Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3 = 0 \\ Z_1 + 2Z_3 = 0 \end{array}\right\} \qquad (27)$$

from which we obtain, immediately,

$$Z_3 = -\tfrac{1}{2}Z_1, \qquad Z_1 = Z_2. \qquad (28)$$

It remains to show that under (28), $B \equiv C$. To see this, substitute (28) in the expressions for B and C, getting

$$B = \begin{vmatrix} R_1 + \tfrac{1}{2}Z_1 & \tfrac{1}{2}Z_1 & -Z_1 \\ -Z_1 & -Z_1 & 2Z_1 + 2R \\ 0 & -R_2 & -R \end{vmatrix}$$

$$C = \begin{vmatrix} R_1 + \tfrac{1}{2}Z_1 & \tfrac{1}{2}Z_1 & -Z_1 \\ \tfrac{1}{2}Z_1 & R_2 + \tfrac{1}{2}Z_1 & -Z_1 \\ -Z_1 & -Z_1 & 2Z_1 + 2R \end{vmatrix}.$$

In the expression for C, interchange the second and third rows, getting

$$C = -\begin{vmatrix} R_1 + \tfrac{1}{2}Z_1 & \tfrac{1}{2}Z_1 & -Z_1 \\ -Z_1 & -Z_1 & 2Z_1 + 2R \\ \tfrac{1}{2}Z_1 & R_2 + \tfrac{1}{2}Z_1 & -Z_1 \end{vmatrix}.$$

Next multiply the elements of the second row by $\tfrac{1}{2}$ and add the results respectively to the elements of the third row, getting

$$C = -\begin{vmatrix} R_1 + \tfrac{1}{2}Z_1 & \tfrac{1}{2}Z_1 & -Z_1 \\ -Z_1 & -Z_1 & 2Z_1 + 2R \\ 0 & R_2 & R \end{vmatrix}.$$

Finally, change the signs of the elements of the last row, to get

$$C = \begin{vmatrix} R_1 + \tfrac{1}{2}Z_1 & \tfrac{1}{2}Z_1 & -Z \\ -Z_1 & -Z_1 & 2Z_1 + R \\ 0 & -R_2 & -R \end{vmatrix} \equiv B$$

which was to be proved.

Theory of Antennas of Arbitrary Size and Shape*

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Summary—*In this paper there are presented (1) a quite general method of antenna analysis; (2) a physical picture of transmission phenomena in antennas, based on this method; and (3) an expression for the input impedance of antennas of any shape, whose transverse dimensions are small compared with the wavelength. In a brief historical sketch of the antenna problem the factors which must be taken into consideration in solving the problem are discussed.*

While in ordinary transmission lines the voltage is proportional to the charge, this is not the case in antennas. The explanation lies in the fact that antennas are multiple *transmission lines (like wave guides) and* not *simple, that is, single-mode transmission lines. Our present theory is based on the voltage-current equations since these appear to be considerably simpler than charge-current equations. The latter are considered only briefly.*

In the absence of dissipation and in so far as the total voltage wave and the "principal" current wave are concerned, radiation is strictly an end effect. In so far as the total current and the total charge waves are concerned, radiation effects are distributed (nonuniformly) along the entire antenna.

In the first approximation, regardless of the shape of the wire the charge is proportional to the voltage and waves are sinusoidal, the current wave having nodes while the voltage wave and the charge wave antinodes at the ends of the antenna. The second approximation depends on the shape of the longitudinal cross section of the antenna as well as on the size of the transverse cross section.

Our analysis is based on Maxwell's equations but the final results are quite simple and the physical picture growing out of this mathematics is attractive to an engineer. It is permissible to think that a wave emerging from a generator in the center of an antenna is guided by an antenna until it reaches its "boundary sphere" passing through the ends of the antenna and separating the antenna region from the external space; at the boundary sphere some energy passes into the external space and some is reflected back—a situation existing at the juncture between two transmission lines with different characteristic impedances. We may also think of the antenna as the wall of an electric horn with an aperture so wide that one can hardly see the horn itself—just like a Cheshire Cat: only the grin can be seen. In fact, the mathematics that we use is that appropriate to wave guides and electric horns.

The antenna problem is stated in Section I and its history is briefly discussed in Section II; Section III contains a summary and a discussion of the results for antennas with uniformly distributed capacitance (conical antennas); Section IV is devoted to antennas with non-uniformly distributed capacitance; Section V presents a derivation of the formulas contained in Section III; Section VI reviews the induced-electromotive-force method of computing radiation and its use in the present problem; Section VII is devoted to the current-charge equations; Section VIII is devoted to wave propagation along parallel wires; in Section IX an expression is given for the impedance of an infinitely long cylindrical wire; and Section X deals with an approximation needed in our discussion of the problem.

I. INTRODUCTION

Two Problems

IT IS beyond the scope of this paper to discuss adequately prior work on radiation from conductors of finite length and only a few representative papers will be cited. Two problems have presented themselves. In Problem A the current distribution in an antenna is given and it is required to find the field and, hence, the external electromotive forces needed to produce the given current distribution. Problem B is the inverse of A: The distribution of applied or external forces in an antenna is given and it is required to obtain the field and, hence, the current produced by the applied forces.

Problem A has been solved rigorously and completely with the aid of retarded potentials. On the other hand, Problem B presents many difficulties and it is the engineer's hard luck that he happens to be interested in just this problem.

Solution of Problem A Is Useful within Limits

The solution of Problem A is not altogether useless to the engineer. Theory is not the only source of information concerning the current distribution produced by given forces. For example, the current distribution can be determined experimentally, thus, it has been known long ago that on a thin wire the current is distributed almost sinusoidally (Fig. 1) and this fact has been employed to obtain approximately the radiated power, the input impedance, and the field. If the length of the wire is in the neighborhood of one half of the wavelength (or one quarter of λ if the ground takes the place of the other half), the results are fairly satisfactory from the practical point of view.[1] But when the length becomes equal to 1 wavelength, then a more accurate solution becomes necessary. The theoretical radiation pattern may be still good enough (except in the former "null directions") but the input
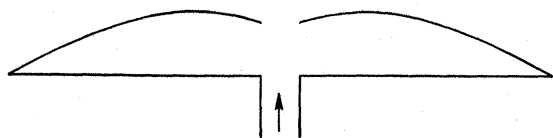

Fig. 1—Current distribution in an infinitely thin perfectly conducting antenna of any shape is sinusoidal with current nodes at the ends of the antenna. This distribution may be taken as the first approximation for thin antennas.

impedance is computed to be infinite. While infinity may be regarded as a legitimate first approximation to a large quantity, it is a useless approximation. For free-space antennas of length $n\lambda$, practical applications require a second approximation and this, in so far as theory is concerned, leads us back to Problem B.

II. History of Problem B and General Comments on Methods for Its Solution

Two Methods of Approach and an Unproved "Proposition α"

Broadly speaking, we may distinguish between two methods of approach to the solution of Problems B. Some writers[2-6] have based their work on Maxwell's equations, that is, on the electromagnetic laws that are considered well established; while, other writers[7-12]

repulsed by the complexity of such methods, ventured to start from a new and unproved premise. This new Proposition α may be formulated as follows: power losses due to radiation produce the same effect on transmission of electric waves along conducting wires as losses due to dissipation. This premise is inconsistent with Maxwell's equations (see Section VIII); and yet, with its aid some approximate results have been obtained. We shall return to this point later.

Spheroidal Antennas

Those writers who prefer "safe" premises as a point of departure have to look for some particular shape of conductors which would lend itself more readily to mathematical treatment. Spheroids have been the first to attract attention. Abraham[2] treated *free* oscillations on a perfectly conducting thin prolate spheroid and obtained an expression for the resonant frequencies or wavelengths. Recently, Page and Adams[5] and, then, Ryder in his as yet unpublished thesis, have dealt with *forced* oscillations on spheroids.

Slow Convergence off Resonance

There is a weakness inherent in the method used by Page and Adams. This method employs spheroidal harmonics and off resonance leads to complicated and slowly converging series. In their numerical computations, the authors limited themselves to spheroids about one-half wavelength long.[13]

The Importance of Shape

An important point, however, is that while in the first approximation the current distribution is independent of the size of the transverse cross section of the antenna and of the shape of the longitudinal cross section and is sinusoidal for all conductors, in the second approximation the current distribution depends on both factors. It was Rayleigh[14] who, in discussing Abraham's[2] and Pollock's[15] papers, was the first to point out that resonant frequencies of finite wires are independent of the shape of the wires in the first approximation but not in the second. Rayleigh's

[1] There may be occasions, of course, when a more accurate solution would be desirable and, certainly, no one is likely to object to having one, provided it is simple enough.

[2] M. Abraham, "Die electrischen Schwingungen um einen stabförmigen Leiter, behandelt nach der Maxwell'schen Theorie," *Ann. der Phys.*, vol. 66, pp. 435–472, 1898.

[3] L. V. King, "On the radiation field of a perfectly conducting base insulated cylindrical antenna over a perfectly conducting plane earth, and the calculation of radiation resistance and reactance," *Phil. Trans.*, ser. A, pp. 381–422; November 2, 1937.

[4] G. Mie, "Elektrische Wellen an Zwei Parallelen Drahten," *Ann. der. Phys.*, p. 248, 1900.

[5] Leigh Page and Norman I. Adams, Jr., "The electrical oscillations of a prolate spheroid, Paper I," *Phys. Rev.*, pp. 819–831; May 15, 1938.

[6] John R. Carson, "Propagation of periodic currents over non-uniform lines," *Electrician*, pp. 272–273; March 4, 1921.

[7] C. P. Steinmetz, "The general equations of the electric circuit—III," *Proc. A.I.E.E.*, p. 255, 1919.

[8] Ronold King, "Telegraphist's equations at ultra-high frequencies," *Physics*, pp. 121–125; April, 1935.

[9] P. O. Pedersen, "Radiation from a vertical antenna over flat perfectly conducting earth, *Ingenirvidenskabelige Skrifter*, ser. A, Nr. 38, 1935.

[10] E. Siegel und J. Labus, "Scheinwiderstand von Antennen," *Hochfrequenz. und Elektroakustik*, Bd. 43, pp. 166–172, 1934.

[11] J. Labus, "Rechnerische Ermittlung des Impedanz von Antennen," *Hochfrequenz. und Elektroakustik*, Bd. 41, pp. 17–23, 1933.

[12] L. J. Chu and J. A. Stratton, "Forced oscillations of a prolate spheroid," *Jour. Appl. Phys.*, pp. 241–248; March, 1941.

[13] Since completion of this paper, L. J. Chu and J. A. Stratton[12] have published a comprehensive discussion of forced oscillations on prolate spheroids and dealt with the conditions off resonance as well as near resonance.

[14] Lord Rayleigh, "On the electrical vibrations associated with thin terminated conducting rods," *Phil. Mag.*, pp. 104–107; July, 1904.

[15] J. A. Pollock, "A comparison of the periods of the electrical vibrations associated with simple circuits," *Phil. Mag.*, pp. 635–652; June, 1904.

conclusion is borne out by Englund's experiments[16] and by our calculations. In fact, not only the resonant frequencies but other quantities as well are affected, in the second approximation, by the size of the transverse cross section of the antenna and the shape of the longitudinal cross section; consequently, a way must be found to take these factors into consideration.

Cylindrical Antennas

A different method was chosen by King.[3] Starting from an integral equation, he obtained the second approximation to the solution of Problem B for a thin cylindrical wire and his work could be extended to wires of other shapes. If anything, this method is more complicated than the one employed in the case of spheroidal antennas and it does not lend itself to any simple physical interpretation; one just has to take the final quantitative results. On the other hand, the calculations are carried out for cylindrical antennas which are of greater practical interest than spheroidal antennas.

Our Method and Its Advantages

We also start with Maxwell's equations but choose *conical* conductors. There are several advantages to this choice. In the first place, the functions to which one is naturally led represent *waves* on the wire rather than *free oscillations*. Consequently, the conditions existing off resonance can be studied just as readily as those near resonance. Furthermore, this means that no complications will arise if we break the wire at some point and insert a resistance, or any impedance, for that matter.

The difference between our method of treating the conical wire and the conventional method of treating the spheroid is precisely the difference existing between two possible methods of dealing with a finite section of an ordinary transmission line. On one hand, the voltage and the current in such a section can always be represented as the result of interference of progressive waves traveling in opposite directions; and, on the other hand, the same quantities can be represented in terms of "harmonics" corresponding to natural oscillations in the section of the line. The first method is so much simpler than the second, which gives the results as infinite series of partial fractions, that probably only few are even aware that the solution could have been found by the second method in the first place.

The Shape of Conductors May Be Taken into Account

Another advantage in our choice of conical shape is that in the second approximation the effects of the shape of the conductor become separated from the "end effect" or radiation.[17] Consequently, the equations developed for conical wires can be amended to take care of the "shape effect."

[16] C. R. Englund, "The natural period of linear conductors," *Bell Sys. Tech. Jour.*, pp. 404–419; July, 1928.

Antenna as a Transmission Line

Finally, our method turns out to be consistent with a physical picture which is rather attractive to the engineer. Let us suppose that a wire is energized at the center. A spherical wave emerging from the generator is guided by the antenna until it reaches the limit of the "antenna region," that is, the sphere passing through the ends of the antenna; there, some of the energy passes into the outer space and some is reflected back, a situation existing when one transmission line is joined to another.

Antenna as an Electric Horn

We may also think of the wire as the wall of an electric horn with an aperture so wide that one can hardly see the horn itself. In fact, the mathematical analysis
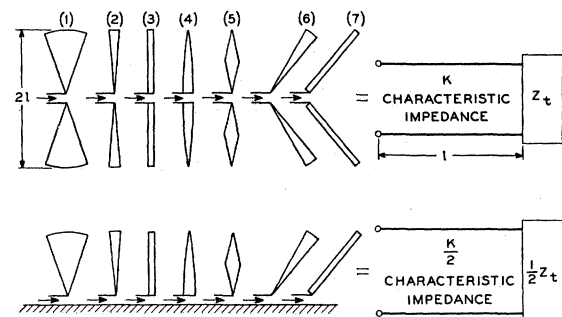

Fig. 2—The input impedance of a conical antenna of any size is equal to the input impedance of a uniform transmission line with a certain "output" impedance Z_t. The input impedance of a thin antenna of any shape may be similarly represented, except that the characteristic impedance will be variable.

used by us is precisely the analysis appropriate to wave guides and electric horns. We end up with a picture of the antenna as a transmission line (Fig. 2) whose output impedance Z_t represents the end effect. The real part R_t of Z_t represents radiation and should properly be called the "radiation resistance"; it is the resistance of the outer space as seen from the ends of the antenna. Unfortunately, this name has been generally given to a quantity which turns out to be equal to K^2/R_t.

Methods Based upon an Unproved Hypothesis

The idea that the ordinary transmission-line theory could be amended to take care of radiation effects had occurred to Steinmetz[7] as long ago as 1919, in connection with his studies of electric waves on parallel wires. He failed to obtain correct results because twice he was deceived by physical intuition. In the first place, he believed that radiation losses and dissipation losses produce the same effect on transmission of waves. In effect this was a new proposition, which we have called Proposition α, that could be inconsistent with established laws and thus he was taking chances. It is perfectly true that each element of current-carrying wire

[17] From the point of view developed in this paper there is no difference between the "end effect" and radiation.

radiates power; but at the same time it receives equal power from the surrounding medium due to the action of other elements plus power that is dissipated in that element.[18] In such circumstances, it is impossible to decide in advance just how radiation affects the current and voltage distribution on the wire.

The Hypothesis Is Wrong

In Section VIII there is given a simple and straightforward proof that the *voltage across a pair of parallel perfectly conducting wires is not attenuated*. This proves that Proposition α is not true as it stands. Nevertheless, we shall have an occasion to point out that there is some truth in this proposition.

Effect of Phase Velocity on Radiation

This brings us to another point. Proposition α has nothing to do (at least directly) with the very large value that Steinmetz obtained for the total radiation from a parallel pair. He used the same method which had been used before and which has been used since for approximate solutions of similar problems; but he apparently thought that since the phase velocity of waves along the line was very high he could make it infinite, and assume, in making calculations of radiation, that the currents along the entire line were in phase. Thus, he found that the radiated power was proportional to the length of the line while in reality the power radiated by a long line is independent of its length.[4,19] This discrepancy is due entirely to the effect of phase velocity of the current waves along the line and has nothing to do with the manner in which radiation affects the current distribution itself. For a discussion of other aspects of the problem of radiation from parallel wires the reader is referred to Carson.[20]

Single Wires

Pedersen[9] and also Siegel and Labus[10,11] made use of Proposition α in their equations for a single wire but they based their computations on more nearly the actual current distribution in the wire. Thus they computed an approximately correct total radiated power and only then they postulated the character of its distribution.

Pederson has tried two different hypotheses. First he assumed that radiation loss is concentrated at the current antinode and then he assumed it to be distributed uniformly along the antenna. The current distributions calculated on either of these two assumptions turned out to be nearly the same and checked fairly satisfactorily with a measured current distribution.

[18] If the wire is perfectly conducting, the tangential electric intensity must vanish at the surface of the wire and the flow of power from the wire or to the wire is 0. If the wire is imperfectly conducting the flow of power is into the wire and not out of it.

[19] John R. Carson, "Radiation from transmission lines," *Jour. A.I.E.E.*, p. 789; October, 1921.

[20] John R. Carson, "The guided and radiated energy in wire transmission," *Jour. A.I.E.E.*, pp. 906–913; October, 1924.

Naturally, the calculated expression for voltage distribution must necessarily be wrong (Section VIII); but no voltage measurements have ever been made. Furthermore, for comparatively short antennas the current distribution does not markedly depend on just where the power is lost (see Section X).

Whatever may be said about the method, Pedersen succeeded in obtaining better approximations to the solution of the antenna problem than the ones available at that time.

III. Antennas with Uniform Characteristic Impedances—General Discussion and Summary

Perfect Conductivity

Unless otherwise specified all conductors are assumed to be perfect. This assumption simplifies the mathematics and separates the effect of radiation on transmission of waves along an antenna from the effect of dissipation. In the first approximation it is reasonable to suppose that the two effects are independent and can be superimposed.

Transmission Modes

Until recently whenever one thought of electric waves guided by a pair of parallel wires or by coaxial cylindrical conductors, one was apt to visualize a picture of electric lines of force extending from one conductor to the other and lying in planes perpendicular to them, that is, in equiphase surfaces. One was conscious that near the ends of the conductors the field was somewhat warped; but one felt that the end effect was small and could be ignored. Thus one was concerned with one configuration of lines of force, with one propagation constant, with one phase velocity, with one characteristic impedance, and with one pair of transmission equations, that is, with one *transmission mode*. A transmission line with a single transmission mode will be called a *simple* transmission line.

But physical transmission lines are multiple. They are capable of guiding many types of waves, with different configurations of lines of force, with different propagation constants and with different characteristic impedances. Recently, this fact has burst into prominence in connection with the waves in hollow metal tubes; but the conventional lines are also multiple lines. The only reason why in the past they were regarded as simple lines is due to the fact that their transverse dimensions were so small compared with the wavelengths in which engineers happened to be interested that only the first transmission mode was quantitatively significant.

This situation may be made clearer by an analogy with an electric circuit comprised of a physical resistor, a physical inductor, and a physical capacitor in series. Such a circuit is usually regarded as a *simple* electric circuit, with one natural frequency or, taking the

decrement into account, with one "natural oscillation constant." In reality the circuit can oscillate in infinitely many modes; it is only because the first natural frequency is very much lower than the rest that in ordinary applications the circuit behaves as if it were a simple circuit.

The most prominent transmission mode of a given line will be called the "principal" or the "dominant" mode.

Principal Waves Guided by Two Coaxial Cones

Let two coaxial conical conductors, having a common axis (Fig. 3), be energized at the common apex. Intuitively one feels that if the cones were of infinite length, the wave would be such that the lines of elec-
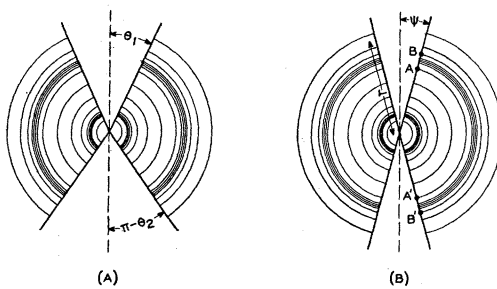
Fig. 3—Cross sections of infinitely long conical conductors and electric lines of force for principal waves.

tric force would follow the meridians of spheres concentric with the apex of the cones. There is indeed a wave, namely, the transverse electromagnetic spherical wave (TEM) for which this is true[21]; this wave turns out to be the most prominent and we shall call it the principal wave.

If we imagine a homogeneous spherical conductor coinciding with some equiphase surface, then the outward-bound progressive wave will be reflected from it. Electric lines, however, will still follow the meridians.

Equations for Principal Waves

Even from elementary considerations it is obvious that in so far as principal waves are concerned, the double cone is a *uniform* transmission line. Consider, for example, the capacitance per unit length AB (Fig. 3 (B)). The length of the lines of electric force and the circumference of the conductor are both proportional to the distance r from the apex; hence, the capacitance remains unchanged.[22]

Principal waves are just as easily treated rigorously.[21] The series inductance L and the shunt capacitance C per unit length and the characteristic impedance K are found to be (for the double cone in Fig. 3(B))

$$L = \frac{\mu}{\pi} \log \cot \frac{\psi}{2}, \qquad C = \frac{\pi \epsilon}{\log \cot \dfrac{\psi}{2}},$$

$$K = \frac{\eta}{\pi} \log \cot \frac{\psi}{2} = 120 \log \cot \frac{\psi}{2}, \qquad (1)$$

where ψ is the angle of the cone.[25] The phase velocity of these waves is equal to that of light.

The principal voltage and the principal current can then be expressed in the following general form

$$V_0(r) = V_0^+ e^{-i\beta r} + V_0^- e^{i\beta r}, \qquad \beta = \frac{\omega}{v} = \frac{2\pi}{\lambda},$$

$$I_0(r) = I_0^+ e^{-i\beta r} + I_0^- e^{i\beta r},$$

$$V_0^+ = K I_0^+, \qquad V_0^- = - K I_0^-. \qquad (2)$$

The voltage between the corresponding points A and A' is defined here as the line integral of the electric intensity along any curve joining A and A' and lying completely in the equiphase surface passing through A and A'. This definition is in keeping with the usual definition of the voltage across a pair of parallel wires. This voltage is difficult to measure except near the origin and may be regarded as an auxiliary variable that helps us to find measurable quantities such as the input impedance, current distribution, etc.

Fig. 4 is a graph of K as a function of the reciprocal angle. For cones of small angle, $1/\psi$ is approximately
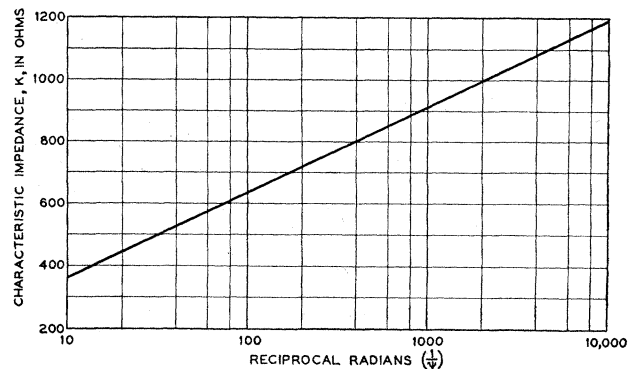

Fig. 4—The characteristic impedance of conical antennas of types shown in Fig. 3(B) as a function of the reciprocal cone angle $\psi = a/l$, where l is the length of the cone and a is its maximum radius.

equal to r/ρ, where ρ is the radius of the cone at distance r from the apex, and

$$K = 120 \log \frac{2}{\psi} = 120 \log \frac{2r}{\rho}. \qquad (3)$$

When this ratio r/ρ is 100, then $K=635$ ohms; and for $r/\rho = 1000$, $K=913$ ohms. It is hardly necessary

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[21] S. A. Schelkunoff, "Transmission theory of spherical waves," *Trans. A.I.E.E.*, pp. 744–750, 1938.

[22] Excellent elementary discussions of principal waves on a double cone may be found in a paper by Howe[23] and on page 183 of a paper by Carter.[24]

[23] G. W. O. Howe, "The nature of the electromagnetic waves employed in radio telegraphy and the mode of their propagation," *Elec. Rev.*, pp. 486–489; September 26, 1913.

[24] P. S. Carter, "Simple television antennas," *RCA Rev.*, pp. 168–185; October, 1939.

[25] The quantity η is the intrinsic impedance of the medium between the cones; for air η is approximately equal to $120\,\pi$ or 377 ohms.

to point out that K varies much slower than the ratio r/ρ.

Standing Principal Waves

If at $r = l$ we assume a reflecting sphere, then the amplitudes of progressive waves in (2) will be equal and standing waves will result. For a perfectly conducting sphere at $r = l$, we shall have

$$I_0(r) = I_0 \cos \beta(l - r),$$
$$V_0(r) = V_0 \sin \beta(l - r), \qquad V_0 = iKI_0. \qquad (4)$$

These are the equations for a spherical cavity resonator and are of no direct interest in our present problem.

If the sphere were a perfect "magnetic" conductor, then the current $I_0(l)$ would have to vanish instead of the voltage and the voltage-current equations would be

$$I_0(r) = I_0 \sin \beta(l - r),$$
$$V_0(r) = V_0 \cos \beta(l - r), \qquad V_0 = -iKI_0. \qquad (5)$$

We know from experience that the current distribution in a thin wire is approximately that given by (5). This suggests that the impedance of free space as seen from the "output" ends of the antenna is so high compared with K that an almost complete reflection takes place. Later in this paper we shall actually prove that as K tends to infinity, the current and voltage distributions on a conical wire, on any wire for that matter, approach (5).

If the spherical sheet at $r = l$ had some finite impedance, we should have

$$I_0(r) = I_0 \sin \beta(l - r) + I^0 \cos \beta(l - r),$$
$$V_0(r) = V_0 \cos \beta(l - r) + V^0 \sin \beta(l - r),$$
$$V_0 = -iKI_0, \qquad V^0 = iKI^0. \qquad (6)$$

The input and the output impedances would then be

$$Z_i = \frac{V_0(0)}{I_0(0)} = K \frac{-iI_0 \cos \beta l + iI^0 \sin \beta l}{I_0 \sin \beta l + I^0 \cos \beta l},$$
$$Z_t = \frac{V_0(l)}{I_0(l)} = \frac{V_0}{I^0} = -iK \frac{I_0}{I^0}. \qquad (7)$$

Sphere of Discontinuity

Equations (6) would have represented the actual conditions in a conical antenna (Fig. 5), were it not for the fact that the space outside the boundary sphere S is a multiple transmission line with a set of transmission modes different from that in the antenna region. In particular, in free space there is no transmission mode which is even similar to the principal mode just discussed by us. For the latter, the field concentration near the conductors (if they are thin) is exceedingly high, because the conductors can support high currents quite readily. On the other hand, all radial currents in dielectrics are comparatively feeble and the magnetic

intensity near the $A - B$ axis outside S must be very small.

Hence the energy carried by the principal wave from 0 to the boundary sphere S must travel thenceforward in different transmission modes and besides an ordinary reflection of the principal wave, secondary waves in
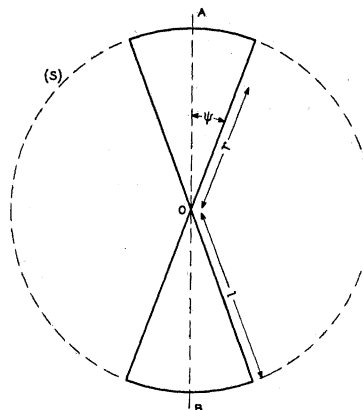

Fig. 5—The cross section of a conical antenna of length l and of the "boundary sphere" S.

the antenna region will be generated to match the field outside S. This is our picture of how the "end effect" comes into being. "Radiation" is one part of this end effect; the reactive field associated with the secondary waves is the other part. The "sphere of discontinuity" is the "aperture" of the antenna regarded as an electric horn.

Free-Space Transmission Modes

Before considering secondary transmission modes in the antenna region, we shall review briefly the modes of transmission in free space. The principal mode in free space is characterized by electric lines of force whose shape is suggested in Fig. 6(A). The energy emitted by a very small doublet travels outward in this mode. The electric field has two components, the radial component E_r and the meridian component E_θ. The
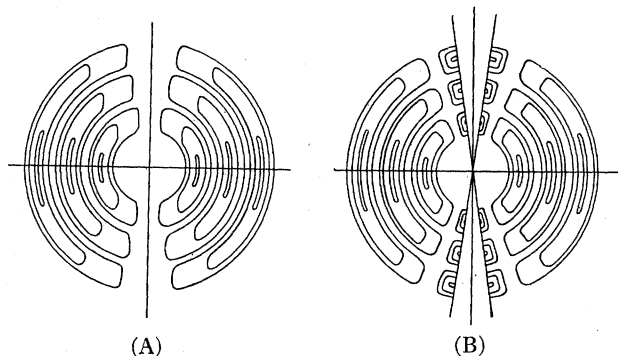

(A)   (B)

Fig. 6—Electric lines for the first-order transverse magnetic spherical waves: (A) lines in free space; (B) lines in the presence of two coaxial conical conductors.

former is proportional to $\cos \theta$ and the latter to $\sin \theta$, where θ is the angle with the axis of the wave. The radial displacement current flows in one direction in the "northern" hemisphere and in the opposite direction in the southern. The two hemispheres play parts

of the two conductors in a transmission line. Of course, the radial displacement current is not distributed uniformly within each hemisphere; the current density is maximum along the axis of the wave. We should note, perhaps, that while E_r varies ultimately as r^{-2}, where r is the distance from the doublet, the radial current density per unit solid angle and hence the total radial current in each hemisphere are independent of the distance, except for the phase factor $e^{-i\beta r}$.

The lines of electric force corresponding to the second transmission mode are shown in Fig. 7(A). Higher transmission modes will have still more sets of closed loops. Mathematically all these modes are represented by zonal harmonics; the radial electric intensity is proportional to $P_n(\cos\theta)$ and the meridian intensity to $(d/d\theta)P_n(\cos\theta)$.
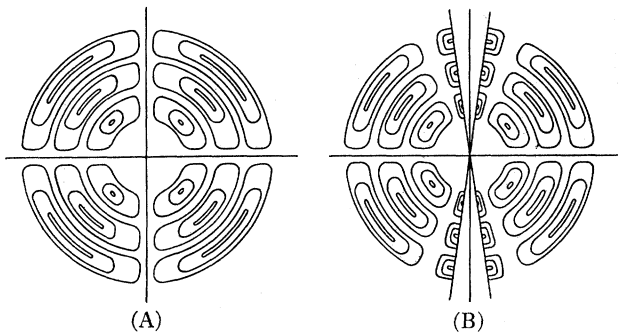

(A)   (B)

Fig. 7—Electric lines for the second-order transverse magnetic spherical waves: (A) lines in free space; (B) lines in the presence of two coaxial conical conductors.

Any field outside the antenna region, having circular symmetry, can be represented as the resultant of a number of waves traveling in the above-discussed transmission modes.

Secondary Transmission Modes in the Antenna Region

Suppose we have a free-space wave like that shown in Fig. 6(A) and suppose we insert a thin conical conductor coaxial with the axis of the wave (Fig. 6(B)). Electric lines of force must be perpendicular to the conductor; but since the meridian intensity near the axis is small, we do not expect a very radical change in the configuration of the lines of force and we expect that this configuration will resemble the one shown in Fig. 6(B). This is indeed the case. The major difference is that in the presence of the conical conductor the radial electric intensity (and hence the radial current density) is highest not at the surface of the conductor but at some distance from it. Right at the surface of the cone, the radial intensity vanishes. As the angle of the cone gets smaller, the field distribution becomes more nearly like free-space field distribution except in the ever-diminishing region adjacent to the cone.

Fig. 7(B) shows how the second[26] mode in the antenna region is related to the second mode in free space. Incidentally, if the antenna is energized at its

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[26] It is really the third if we count the principal; but it is more convenient to designate the principal mode in the antenna region as the 0th mode.

center, this mode and all other even modes do not appear. This is because the conduction currents associated with them flow in opposite directions in the two halves of the antenna, while the currents produced by the generator must flow in the same direction.

The Total Voltage Associated with Any Secondary Wave in the Antenna Region Is Zero

As in the case of the principal wave we define the voltage between two points on the upper and lower cones as the line integral of the electric intensity along any path joining the point of the upper cone to the point of the lower cone, provided the path is situated in the equiphase circuit passing through the two points.[27] It is shown in Section V that for all secondary waves in the antenna region this voltage is equal to zero

$$V_m(r) = 0. \tag{8}$$

Conduction Currents Accompanying Secondary Waves Vanish at the Origin

It is shown in Section V that the conduction current accompanying any secondary wave vanishes at the origin

$$I_m(0) = 0. \tag{9}$$

Near the origin $I_m(r)$ varies as $r^{m+1+\Delta}$, where, approximately, $\Delta = 120/K$. Thus the effect of the secondary current waves on the total current is rather unimportant near the origin, but becomes more pronounced near the output terminals of the antenna.

Voltage-Current Equations

We can now write the complete voltage-current equations for a perfectly conducting transmitting antenna energized at its center in the following form:

$$V(r) = V_0(r), \qquad I(r) = I_0(r) + \bar{I}(r), \tag{10}$$

where the principal voltage-current waves are given by (6) and the total secondary current $\bar{I}(r)$ is the sum of odd secondary current waves

$$\bar{I}(r) = I_1(r) + I_3(r) + I_5(r) + \cdots, \quad \bar{I}(0) = 0. \tag{11}$$

The total voltage wave consists of just two principal waves, of which the second represents the effect of incomplete reflection at the boundary sphere S. As the result of this incomplete reflection, the voltage maximum does not occur at $r = l$. This is the only effect of radiation on the voltage wave; no "attenuation" is introduced into the voltage wave.

The total current wave, on the other hand, is affected more radically.

Imperfectly Conducting Antennas

It will be seen that $\bar{I}(r)$ is rather small compared with $I_0(r)$ and it is natural then to assume that the

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[27] If the points are not situated in the same equiphase surface, then the voltage between them is not defined.

imperfect conductivity of an antenna will manifest itself largely through the principal wave which will become

$$I_0(r) = -iI_0 \sinh \Gamma(l-r) + I^0 \cosh \Gamma(l-r),$$

$$V_0(r) = -iKI_0 \cosh \Gamma(l-r) + KI^0 \sinh \Gamma(l-r). \quad (12)$$

The propagation constant Γ is given by

$$\Gamma = \frac{R}{2K} + i\beta, \quad (13)$$

where R is the resistance (of both cones) per unit length.

The Charge Is not Generally Proportional to the Voltage

It is evident that for principal waves, the electric charge $q_0(r)$ per unit length is proportional to the volt-
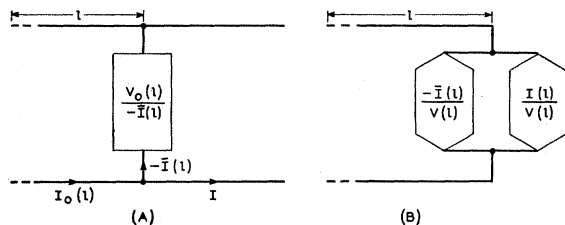

Fig. 8—The relation between principal, secondary, and total currents at the ends of an antenna.

age $V_0(r) = V(r)$. The charge $q_m(r)$ associated with a secondary wave, being proportional to the derivative of the current, does not vanish while the corresponding voltage does; this means that the total charge is not proportional to the total voltage.

In a simple uniform transmission line the charge is proportional to the voltage and it does not matter whether we write the transmission equations in the voltage-current form or in the charge-current form; but these two possible forms are quite different in the case of multiple transmission lines unless one mode predominates over all others. The charge-current equations are more complicated than the voltage-current equations and are here considered (see Section VII) only because of their bearing on the idea that radiation takes place continuously along the antenna. In that section we shall show that the effect of radiation on the distribution of the total charge (but not the total voltage) and the total current in an antenna may be represented by a continuously distributed series resistance and a continuously distributed series inductance in addition to the normal inductance of the antenna regarded as a simple transmission line. This coupled with the fact that for comparatively short antennas the precise distribution of this resistance has but little effect on the current distribution, explains why equations obtained by Pedersen, Siegel, Labus and others turn out to be fairly satisfactory approximations to the antenna solution, provided we replace in their equations V by a quantity proportional to q. On the whole, however, we find the voltage-current equations

much simpler; and in these equations, radiation may be represented as a terminal impedance.

Secondary Waves in an Antenna Affect the Amplitudes of the Principal Waves in a Way an Output Impedance Would

Consider the principal waves in the antenna as given by (6). Substituting from (10) into the expression for the output impedance Z_t and taking its reciprocal, we have

$$Y_t = \frac{1}{Z_t} = \frac{I_0(l)}{V_0(l)} = \frac{I(l)}{V(l)} + \frac{-\bar{I}(l)}{V(l)}. \quad (14)$$

Thus the output admittance consists of two admittances in parallel.

Transmission-line diagrams (Fig. 8) represent the above relationship graphically. The current $-\bar{I}(l)$ is that part of the principal current which is diverted into a shunt admittance. Whenever a capacitor (or any impedor) is shunted across a pair of wires, we can look upon this capacitor as a practical means for producing a local field superimposed upon the normal field surrounding the pair of wires. Broadly speaking the capacitor is an irregularity in the transmission line. Sudden termination of the wires in the antenna case is also an irregularity producing a field which is superimposed upon the normal field of the principal waves guided by the antenna.

The total current $I(l)$ may be different from zero. For instance, if the tops of the conical conductors are large, appreciable current may flow over the edge; or, if the antenna itself is thin, end capacitances may be provided by a number of wires spreading fanwise from the ends of the antenna wires. However, if the cross section of the ends of the antenna is small and no loading is provided, then the total current is zero and

$$I_0(l) = -\bar{I}(l), \qquad Y_t = \frac{-\bar{I}(l)}{V(l)}. \quad (15)$$

This is the case with which we are specifically concerned in this paper.

The Output Impedance of a Transmitting Antenna

In Section V we shall prove that if the characteristic impedance of the antenna is large and if the total current at the ends of the antenna is zero, then we have approximately

$$\frac{I^0}{I_0} = \frac{F(L) - iG(L)}{K}, \quad (16)$$

where the "phase length" of each cone is[28]

$$L = \frac{2\pi l}{\lambda}. \quad (17)$$

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[28] It is hoped that our use of L to designate the phase length and the inductance will not lead to confusion. This use of L in the second sense is only occasional and it will be obvious from the context when it is meant to designate the inductance.

The functions $G(L)$ and $F(L)$ are given by the series

$$G(L) = 30\pi L \sum_{m=0}^{\infty} \frac{4m+3}{(m+1)(2m+1)} J_{2m+3/2}^2(L),$$

$$F(L) = -30\pi L \sum_{m=0}^{\infty} \frac{4m+3}{(m+1)(2m+1)} J_{2m+3/2}(L) N_{2m+3/2}(L). \tag{18}$$

Presently we shall give much simpler expressions for these functions.

Substituting in (7), we have for the output impedance

$$Z_t = \frac{K^2}{G(L) + iF(L)}. \tag{19}$$

It is worth noting that at a distance of $\frac{1}{4}$ wavelength from the terminals, this impedance appears as

$$\frac{K^2}{Z_t} = G(L) + iF(L). \tag{20}$$

The graphs of the real and imaginary parts of this transformed impedance are shown in Fig. 9.

The Input Impedance

The input impedance of the antenna is obtained from the usual transmission-line equations or directly from (7); thus we have

$$Z_i = K \frac{(G + iF) \cos\left(L - \frac{\pi}{2}\right) + iK \sin\left(L - \frac{\pi}{2}\right)}{K \cos\left(L - \frac{\pi}{2}\right) + i(G + iF) \sin\left(L - \frac{\pi}{2}\right)}$$

$$= K \frac{G \sin L + i(F \sin L - K \cos L)}{(K \sin L + F \cos L) - iG \cos L}. \tag{21}$$

Separating the real and the imaginary parts, we obtain

$$Z_i = \frac{G - i\left[\frac{1}{2}K \sin 2L + F \cos 2L - \frac{F^2 + G^2}{2K} \sin 2L\right]}{\sin^2 L + \frac{F}{K} \sin 2L + \frac{F^2 + G^2}{K^2} \cos^2 L}. \tag{22}$$

The Case of an Infinitely Large K

If K is infinitely large, then $I^0 = 0$ and the current distribution in the antenna becomes

$$V(r) = V_0 \cos \beta(l - r),$$
$$I(r) = I_0 \sin \beta(l - r). \tag{23}$$

From (22) we find that the input impedance tends asymptotically to

$$Z_i \to \frac{G(L) + iF(L)}{\sin^2 L} - iK \cot L. \tag{24}$$

Since the input current tends to $I_0 \sin L$, the complex flow of power is

$$\Psi = \frac{1}{2}\left[G(L) + iF(L) - \frac{1}{2}iK \sin 2L\right]I_0^2. \tag{25}$$

The preceding expressions are the first approximations to the antenna equations.

Formulas for $G(L)$ and $F(L)$

The above asymptotic formulas provide a very convenient method for obtaining G and F functions. Since (25) represents the power flow *when the current is sinusoidal* with a node at the end of the antenna, $G(L)$ must be the so-called "radiation resistance" as referred to the maximum current. This resistance is independent of the shape of the antenna and hence is equal to that of a cylindrical wire. The latter can and has been computed either by the Poynting flux method
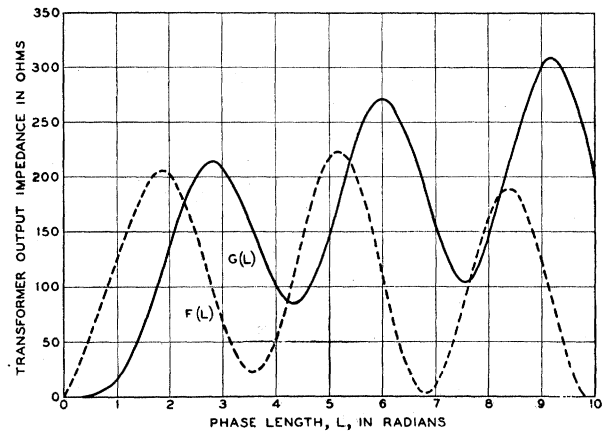

FIG. 9—The real and the imaginary parts of the "transformed" output impedance $K^2/Z_t = G(L) + iF(L)$, where $L = 2\pi l/\lambda$.

or by the induced-electromotive-force method (see Section VI). The latter method can also be used for computing $F(L)$, except that this time it is necessary to make calculations for *conical* antennas. As will be seen in Section IV, antennas of other shapes than conical will have an extra reactive component which must be attributed to the nonuniformity of the line rather than to the end effect.

Thus we have obtained the following expressions

$$G(L) = 60(C + \log 2L - \text{Ci } 2L) + 30(C + \log L - 2 \text{ Ci } 2L$$
$$+ \text{ Ci } 4L) \cos 2L + 30(\text{Si } 4L - 2 \text{ Si } 2L) \sin 2L,$$

$$F(L) = 60 \text{ Si } 2L + 30(\text{Ci } 4L - \log L - C) \sin 2L$$
$$- 30 \text{ Si } 4L \cos 2L, \tag{26}$$

where $C = 0.577 \cdots$ is Euler's constant.

The method just outlined can be used successfully for obtaining second approximations to antenna problems when terminal conditions are other than those considered in this paper.

Resonance

From (22) we observe that the input reactance will vanish when

$$\tan 2L = -\frac{2KF}{K^2 - G^2 - F^2}. \tag{27}$$

Inasmuch as we are concerned here with approximations as far as the first powers of the characteristic admittance $1/K$, we approximate (27) by

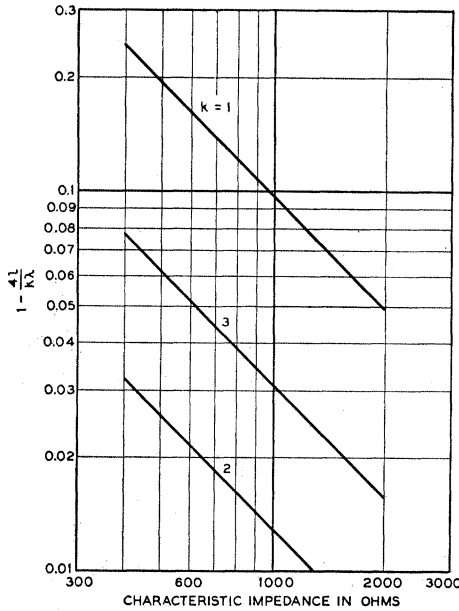$$\tan 2L = -\frac{2F}{K}. \tag{28}$$

Fig. 10—Deviation of the resonant length of conical antennas from $2l = k\lambda/2$.

The solution of the above equation is

$$2L = k\pi - \frac{2F\left(\frac{k\pi}{2}\right)}{K}, \quad k = 1, 2, \cdots. \tag{29}$$

Hence, the input reactance of a conical antenna vanishes when

$$\frac{4l}{k\lambda} = 1 - \frac{2F\left(\frac{k\pi}{2}\right)}{\pi k K} = 1 - \frac{120 \operatorname{Si} k\pi + 60(-)^{k+1} \operatorname{Si} 2k\pi}{\pi k K}. \tag{30}$$
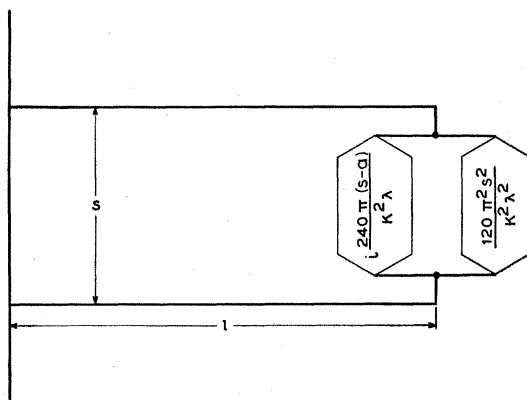

Fig. 11—A Lecher system comprised of two parallel wires, short-circuited at the left end with a metal disk. A parallel combination of two admittance boxes on the right represents the end effect; the conductance box represents radiation and the second box the end capacitance. It is assumed that s is substantially smaller than l.

If it were not for the end effect, the resonant lengths of antennas would be given by $2l = k\lambda/2$. The end effect makes antennas resonate when they are somewhat shorter than $k\lambda/2$. For the first three resonances the shortening effect is shown in Fig. 10. For higher resonances the shortening is substantially equal to $45\lambda/K$, when k is odd, and $15\lambda/K$ when k is even.

In antennas of other shapes than conical, another factor affects resonant lengths. This factor will be considered in Section IV.

Resonance in Lecher Systems

The shortening effect just discussed exists also in Lecher systems and for the same reason. A sudden discontinuity introduces a terminal capacitance. Con-
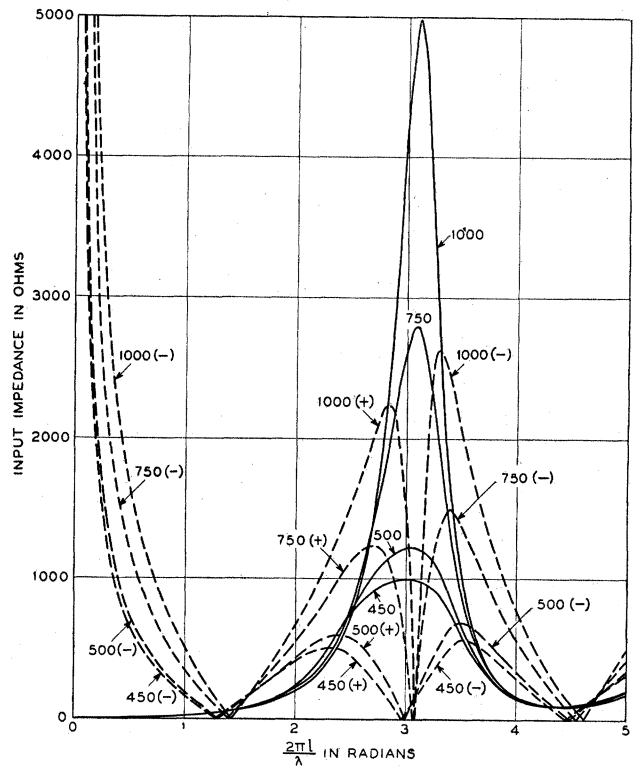

Fig. 12—The input impedance of conical antennas as a function of $2\pi l/\lambda$ and K. Solid curves represent the real component and the dotted curves the imaginary.

sider a Lecher system "short-circuited" with a metal disk at one end and open at the other (Fig. 11). The "open end" is not electrically open and the impedance across it represents the power loss by radiation and the end capacitance. If s is the interaxial separation between the wires and a is their radius, then in the neighborhood of the principal resonance we have approximately

$$C = \frac{120(s-a)}{K^2 v}, \quad G = \frac{120\pi^2 s^2}{K^2 \lambda^2}, \quad K = 120 \log \frac{s}{a}, \tag{31}$$

where v is the velocity of light. These values have been computed by the method of the induced electromotive force from the sinusoidal current distribution, taken as the first approximation to the true distribution.

In this case it is easy to show that for the principal resonance

$$\lambda = 4l + 4vCK. \tag{32}$$

Substituting from (31), we have

$$\lambda = 4l + \frac{480(s - a)}{K}. \tag{33}$$

For this case, Englund[16] has obtained the following relation experimentally

$$\lambda = 4l + 12.4, \tag{34}$$

for wavelengths from 400 to 750 centimeters. In his setup $s = 10.1$ centimeters, $a = 0.635$ centimeter, and
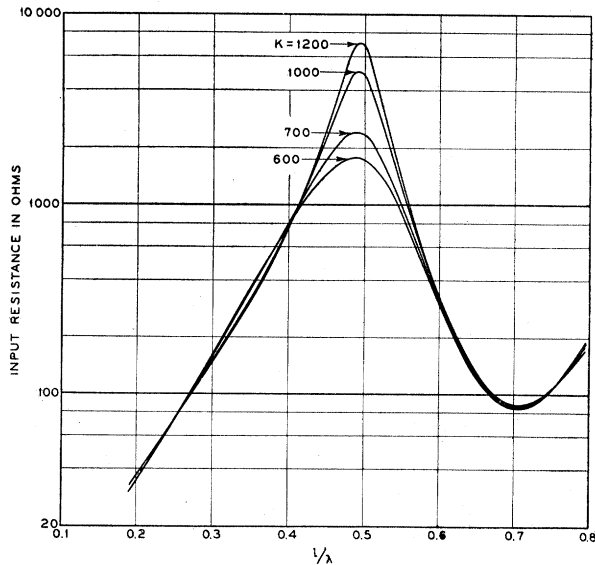

Fig. 13—The input resistance of conical antennas.

$K = 332$ ohms. Substituting these values in (33) we obtain

$$\lambda = 4l + 13.7. \tag{35}$$

Some of the discrepancy between the measured and the calculated values is probably due to the fact that the diameter of the short-circuiting disk was only 15.5 centimeters whereas in computing C we have assumed the disk to be large enough for the current in it to produce the same effect on the Lecher system as the "image" of the system in the disk. Some of the discrepancy (to the extent of a few millimeters) must be ascribed to the second-order errors in the computed value and in measurements. The more precise value of C contains small terms depending on the length l of the Lecher system; but Englund's formula does not include such a term.

Input Resistance and Reactance Curves

The input impedance of conical antennas may be computed from (22). This impedance depends on the characteristic impedance and on the length of the antenna. In Fig. 12 the resistance and the reactance are plotted as functions of $2\pi l/\lambda$, where $2l$ is the length of

the antenna in free space. The characteristic impedance K is the parameter; it is defined by (1) and plotted in Fig. 4. For a vertical antenna of length l
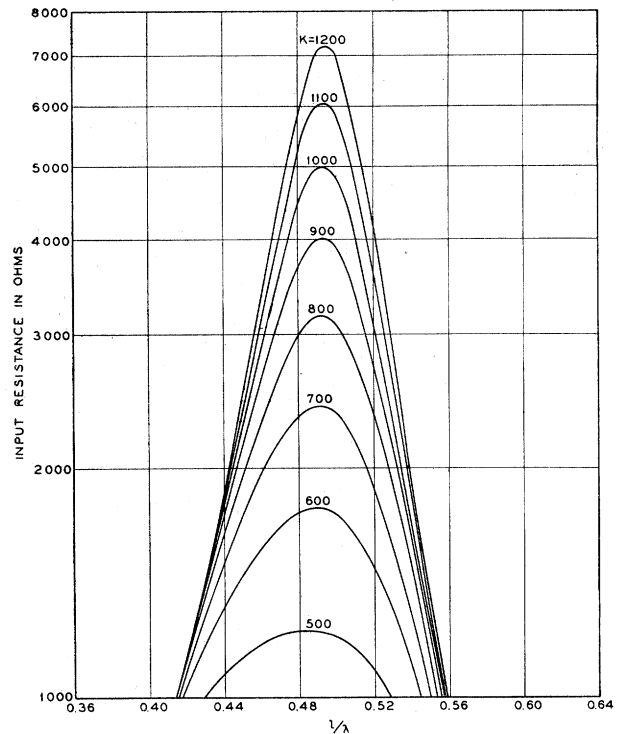

Fig. 14—The input resistance of conical antennas in the neighborhood of the second resonance.

above a perfectly conducting ground we can use the same set of curves and only divide by two the ordinates and the characteristic impedance. For example, the maximum resistance of an antenna in free space, with $K = 1000$, is about 5000; for a vertical antenna of the same size, $K = 500$, and the maximum resistance is 2500.

In Fig. 13 the input resistance alone is shown. It will be observed that the input resistance depends very markedly on K in the region (shown on a larger scale
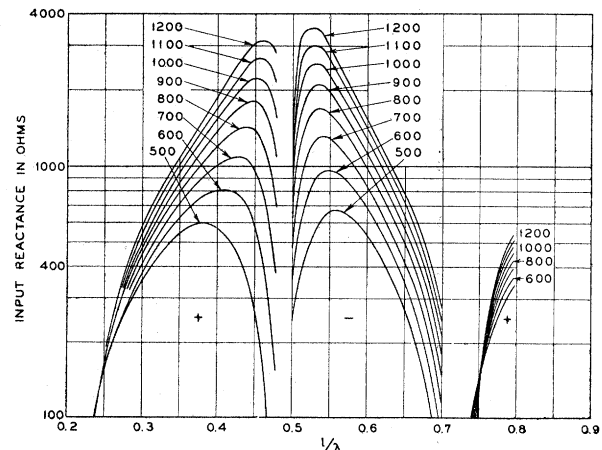

Fig. 15—The input reactance of conical antennas.

in Fig. 14) around the second resonance, or more generally in regions around even resonances. The reactance curves are shown separately in Fig. 15.

Current Distribution

In practice, precise knowledge of current distribution is of lesser importance than knowledge of the input impedance. This is because the directive gain of antennas and their radiation patterns are not very sensitive to the changes in the current distribution. Radiation patterns will be affected seriously only in those directions in which radiation is small.
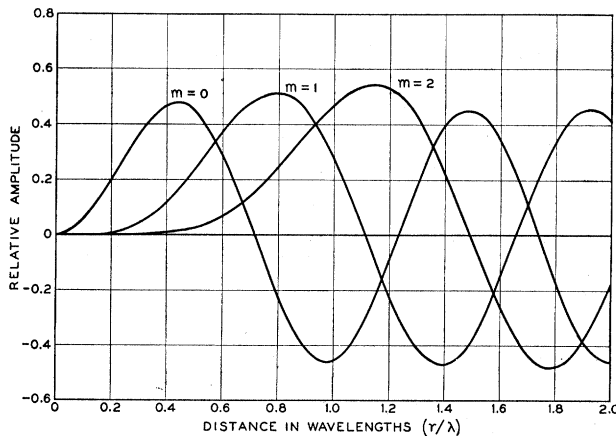

Fig. 16—The amplitudes of the first, third, and fifth ($m = 0$, 1, 2) secondary waves as functions of the distance from the center of the antenna.

The current distribution is given by (10) and (11). In Section V we obtain the following approximate expression for a typical secondary wave:

$$I_{2m+1}(r) = I_0 \frac{30\pi(4m+3)L}{(m+1)(2m+1)K} \left[N_{2m+3/2}(L) + iJ_{2m+3/2}(L) \right]$$

$$\cdot \sqrt{\frac{r}{l}} J_{2m+3/2}\left(\frac{Lr}{l}\right). \qquad (36)$$

First of all let us consider individual secondary waves. Near $r = 0$, the amplitude of the $(2m+1)$st current wave varies nearly[29] as r^{2m+1}. Fig. 16 illustrates the way in which the amplitudes of the 1st, 3rd, and 5th ($m = 0$, 1, 2) vary with the distance from the origin. If this distance is less than $\frac{1}{4}$ wavelength, only the 1st wave is important; at distances smaller than $\frac{1}{2}$ wavelength only the 1st and the 3rd[30] are important.

The maximum amplitudes of secondary current waves depend on the characteristic impedance and on the length of the antenna. Fig. 17 shows actual secondary current waves for $K = 1000$ and $l = \lambda/2$. Of the components in phase (solid curves) with the dominant current $I_0 \sin \beta(l-r)$, only the first is important except at $r = l$; of the quadrature components the first two are sufficient. Both components are inversely proportional to K.

In Fig. 18 the solid curve represents the amplitude of the total current, the dash curve shows the amplitude of the component in phase with I_0, and the dash-

[29] More accurately, the amplitude varies as $(2m+1+120/K)$th power of r.
[30] The even-order waves are absent when a generator is at the center.

dot curve represents the amplitude of the quadrature component. In this figure the current does not quite vanish at the end of the antenna; this is because near the very end of the antenna that part of the total secondary current which is in phase with the dominant current term $I_0 \sin \beta(l-r)$ is determined by a very large number of secondary-current waves and in computing our curve we have taken into account only two. This situation is closely related to very slow convergence of the series representing $F(L)$ in (18) and it is understandable on physical grounds. The field distortion in the immediate vicinity of a sharp end must be much greater than elsewhere and more terms will be needed to represent the field accurately. If we dissolve both the current and the charge near the end of the antenna into two quadrature components, then it becomes evident that the slope of the current curve depends on the charge at the end and must be quite large. Thus near the ends of the antenna, the current approaches zero very abruptly.

In Fig. 19 the total current in the antenna is compared with the principal current. The difference between the real parts is seen to be quite small; but the difference between the imaginary parts is relatively large, except near the center.

Minimum Amplitude

We shall now find the ratio I_{min}/I_{max} of the first minimum amplitude to the first maximum amplitude (counting from the generator). The first minimum is relatively close to the generator where the secondary
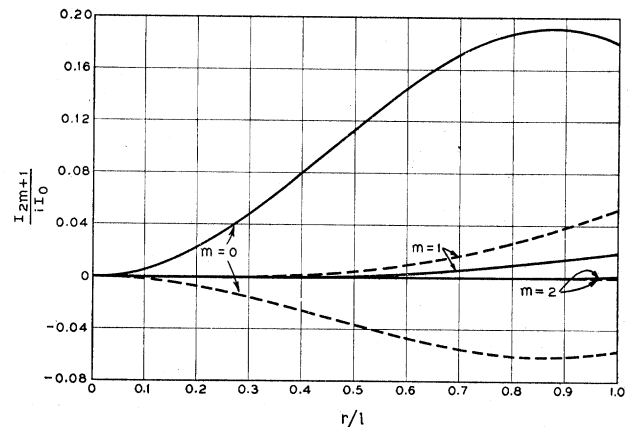

Fig. 17—Secondary current waves for $K = 1000$ and $l = \lambda/2$. The solid curves show the components in phase with the dominant current $I_0 \sin \beta(l-r)$, and the dotted curves show the quadrature components.

current wave is very small. The first maximum is farther away where the secondary current is greater; but there the principal current is large. Thus we can obtain the desired information fairly accurately from the principal current

$$\frac{I_0(r)}{I_0} = \left[\sin \beta(l-r) + \frac{F(L)}{K} \cos \beta(l-r) \right]$$

$$- i \frac{G(L)}{K} \cos \beta(l-r). \qquad (37)$$

The maximum value of this ratio will occur where $\sin \beta(l-r)$ is nearly unity and where $\cos \beta(l-r)$ is, therefore, nearly zero. Hence, the maximum value of (37) is nearly unity. The minimum value will occur where $\cos \beta(l-r)$ is nearly unity. In the vicinity of this point, we can find a value of $\beta(l-r)$ for which the real part of (37) vanishes; thus, G/K is nearly the minimum of $I_0(r)/I_0$ and

$$\frac{I_{\min}}{I_{\max}} = \frac{G(L)}{K} \, . \tag{38}$$

There are no measurements of current distribution in antennas with uniform characteristic impedance. However, it is of interest to compare the measurements on nonuniform antennas with values computed for uniform antennas. In the case of the Copenhagen antenna,[9] a vertical antenna directly above the ground, the *average* characteristic impedance is 540 ohms and $L_{\text{eff}} = 3.64$. Replacing the ground by the image of the antenna, we take the characteristic impedance of the corresponding free-space antenna[31] as 1080 ohms. The ratio $I_{\min}/I_{\max}$ is computed to be 0.131. As nearly as we can read from Pedersen's picture, enlarged by King,[3] the experimental value is 0.132. The ratio calculated by King is approximately $3/34 = 0.0882$; this is lower than the experimental value by about 33 per cent, the difference being considerably larger than the experimental error. We can offer no explanation of this discrepancy. King's method is rigorous; and the ac-
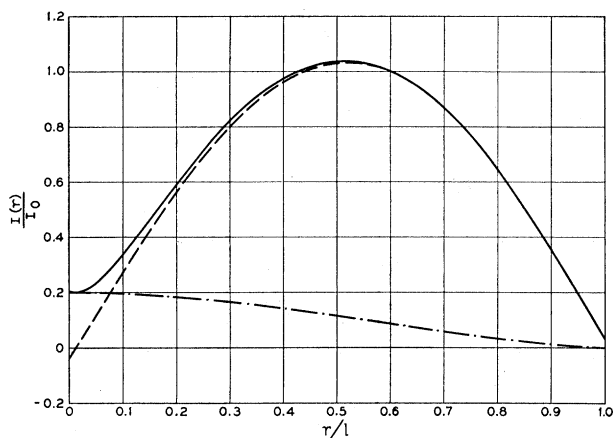

Fig. 18—The total current in the antenna of length $2l = \lambda$; $K = 1000$. The solid curve represents the amplitude of the total current; the dash curve represents the amplitude of the component in phase with I_0 and the dash-dot curve is the amplitude of the quadrature component.

curacy of his approximation would seem to be of the same order of magnitude as ours.

Morrison and Smith[32] have measured the current distribution in a tower 6 feet and 6 inches square and 400 feet long. The diagonal of the cross section is 9.2

[31] See Section IV for a method of computing average impedances.
[32] J. F. Morrison and P. H. Smith, "The shunt-excited antenna," Proc. I.R.E., vol. 25, pp. 673–696; June 1937.

feet and rather arbitrarily we have chosen $2a = 8$ feet as the diameter of an equivalent circular tower in our computation of the average characteristic impedance. Thus we have obtained $K = 516$ ohms (for free space; 258 ohms for the actual tower above the ground). The phase length of the antenna is $L = 3.7$ and $G(L) = 134$.
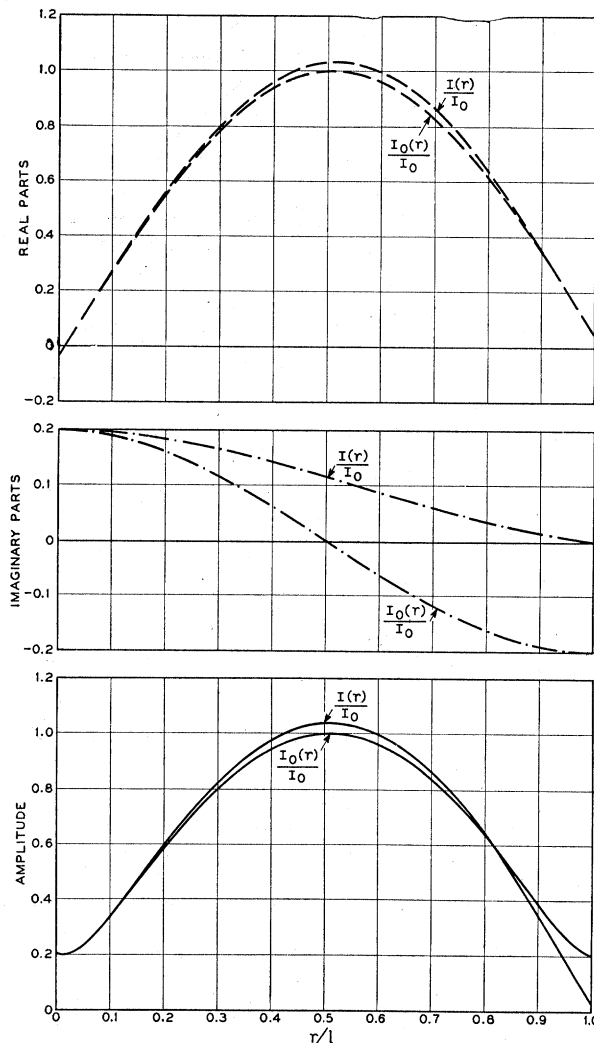

Fig 19—Curves for the total current and the principal current.

The calculated ratio is 0.26 and the measured ratio as read from the picture[32] is $17.5/75 = 0.233$.

Transmitting Antennas Fed at an Arbitrary Point and Receiving Antennas

Most of the foregoing formulas have been derived specifically for free-space antennas fed at the center (or for vertical antennas directly above ground). The method described in Section V is applicable equally well to other cases. Here we shall limit ours to a few general remarks.

If two equal electromotive forces[33] are applied at two points equidistant from the center, then the current distribution will still be given in the form (10) and (11).

[33] By "equal" we mean equal amplitudes and equal phases, so that at all times the forces act in the same direction.

However, the ratio I^0/I_0 will have to be determined anew as a function of the distance from the center as well as a function of K and L.

If two equal but oppositely directed electromotive forces are applied at points equidistant from the center, then none of the current components in (10) and (11) will be present in the new expression for the current. This is because all the terms in (10) and (11) correspond to currents flowing in the same direction in the two halves of the antenna. The new expression will contain its own principal wave, *principal* antisymmetric wave, and a set $I_2(r)$, $I_4(r)$, $\cdots$ of antisymmetric secondary waves.

An electromotive force V applied at some point can always be regarded as the resultant of two pairs of forces $(V/2, V/2)$ and $(V/2, -V/2)$ applied at points equidistant from the center.

The same general method can also be used to develop a theory of "end-fed" wires by considering a single cone, instead of a double cone.

IV. Antennas with Variable Characteristic Impedances

The Problem

Conical antennas support spherical waves regardless of the magnitude of the cone angle and their theory is relatively simple. Antennas of other shapes are definitely more complicated. However, if their transverse dimensions are small enough, they support approximately spherical waves and approximate solutions are rather simple. We obtain these approximations largely on the basis of the physical picture implied by the theory of conical antennas rather than by a direct mathematical analysis.

Principal Waves

Imagine a set of spheres concentric with the generator. Each small segment of the antenna may be regarded as a section of a cone and we may write the following approximate expressions for the distributed series inductance and shunt capacitance per unit length:

$$L = \frac{\mu}{\pi} \log \frac{2}{\psi} = \frac{\mu}{\pi} \log \frac{2r}{\rho},$$

$$C = \frac{\pi\epsilon}{\log \dfrac{2}{\psi}} = \frac{\pi\epsilon}{\log \dfrac{2r}{\rho}}. \qquad (39)$$

The transmission equations for the principal waves will be

$$\frac{dV}{dr} = -i\omega LI, \qquad \frac{dI}{dr} = -i\omega CV. \qquad (40)$$

If $\rho \to 0$, L and C become increasingly more constant as r varies. We expect, therefore, the following asymptotic solutions of (40)

$$V(r) = \sqrt{K(r, \rho)} \left(Ae^{-i\beta r} + Be^{i\beta r} \right),$$

$$I(r) = \frac{Ae^{-i\beta r} - Be^{i\beta r}}{\sqrt{K(r, \rho)}}, \qquad (41)$$

where the characteristic impedance $K(r, \rho)$ is now a slowly varying function of the distance from the generator

$$K(r, \rho) = \frac{1}{\pi} \sqrt{\frac{\mu}{\epsilon}} \log \frac{2r}{\rho} = 120 \log \frac{2r}{\rho}. \qquad (42)$$

It is interesting to note that (41) are exactly the approximate expressions obtained by Brillouin[34] directly from the second-order differential equations resulting from elimination of either V or I from (40).

Average Characteristic Impedances

We shall define the average characteristic impedance K_a as follows

$$K_a = \frac{1}{l} \int_0^l K(r, \rho) dr. \qquad (43)$$

Applying this formula to a cylindrical wire, we obtain

$$K_a = 120 \left(\log \frac{2l}{a} - 1 \right); \qquad (44)$$

this is substantially equal to the characteristic impedance of a cylindrical antenna as given by Pedersen.[9]

For a spheroidal antenna, the average characteristic impedance is

$$K_a = 120 \log \frac{l}{a}, \qquad (45)$$

where "a" is the radius at the base of the antenna.

For an antenna of the shape (5) among those in Fig. 2, the average characteristic impedance is

$$K_a = 120 \log \frac{2l}{a}, \qquad (46)$$

where "a" is the maximum radius of the antenna. These average characteristic impedances are shown in Fig. 20 as functions of l/a.

As K_a approaches infinity, the difference $(K(r, \rho) - K_a)/K_a$ approaches 0 and the transmission line becomes more nearly uniform. Consequently, we may regard the uniform line with the characteristic impedance K_a as a first approximation to the given nonuniform line. The second approximation will contain terms depending on the reciprocal of K_a and these terms are large enough to be of importance in engineering applications. For example, sections of transmission lines with variable $K(r, \rho)$ do not resonate at the same frequencies as uniform sections of equal lengths. The effect is of the same order of magnitude as the effect

[34] J. C. Slater and N. H. Frank, McGraw-Hill Book Company, New York, N. Y., (1933), pp. 147–148.

due to radiation and may either aid or oppose it. We can compute this effect if we find the second approximation to the input impedance of nonuniform lines; but there exists a somewhat simpler method which will be employed in the following section.

Variable Capacitance and Inductance Affect Resonance Conditions in Finite Sections of Nonuniform Transmission lines

Multiplying the first equation in (40) by I^* and the conjugate of the second by V, adding the results and integrating from 0 to l, we have

$$V(l)I^*(l) - V(0)I^*(0) = i\omega \int_0^l [CVV^* - LII^*]dr. \quad (47)$$

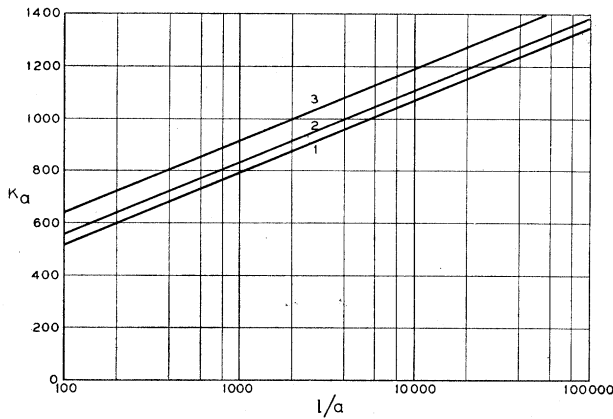If the section of the line is either electrically open or

Fig. 20—The average characteristic impedance: (1) cylindrical antenna, (2) spheroidal antenna, (3) antenna shape (5) shown in Fig. 2.

short-circuited at both ends, the left side of this equation vanishes and we have

$$\int_0^l CVV^*dr = \int_0^l LII^*dr. \quad (48)$$

This is merely an expression of the well-known fact that at resonance the average electric energy and the average magnetic energy are equal. Furthermore, Rayleigh has shown that the first-order errors in distribution of V and I result in second-order errors in the resonant frequencies. Thus, (48) can be used for an approximate computation of resonant frequencies.

First of all, however, we shall obtain two special forms of (48), especially adapted to the case when

$$LC = \frac{1}{v^2} = \text{constant}. \quad (49)$$

Taking this relationship into consideration, we rewrite (40) as follows:

$$I = \frac{iv^2C}{\omega}\frac{dV}{dr}, \qquad V = \frac{iv^2L}{\omega}\frac{dI}{dr}. \quad (50)$$

Substituting from (50) in (48), we have

$$\frac{\omega^2}{v^2} = \frac{4\pi^2}{\lambda^2} = \frac{\int_0^l L\left|\frac{dI}{dr}\right|^2 dr}{\int_0^l L|I|^2 dr} = \frac{\int_0^l C\left|\frac{dV}{dr}\right|^2 dr}{\int_0^l C|V|^2 dr}. \quad (51)$$

The foregoing formulas are exact. In order to obtain approximate formulas, we shall assume sinusoidal voltage-current distributions. Thus, for the first resonance with one end short-circuited and the other open (let the end at $r=0$ be short-circuited and the end at $r=l$ open), we have

$$I(r) = I\cos\frac{\pi r}{2l}, \qquad V(r) = V\sin\frac{\pi r}{2l}. \quad (52)$$

Substituting in (51), we obtain

$$\frac{16l^2}{\lambda^2} = \frac{1-\chi}{1+\chi} = \frac{1+\bar{\chi}}{1-\bar{\chi}}, \quad (53)$$

where

$$\chi = \frac{\int_0^l L\cos\frac{\pi r}{l}dr}{\int_0^l Ldr}, \qquad \bar{\chi} = \frac{\int_0^l C\cos\frac{\pi r}{l}dr}{\int_0^l Cdr}. \quad (54)$$

Similarly, for the first resonance with both ends open, $I(r)=I\sin(\pi r/l)$ and

$$\frac{4l^2}{\lambda^2} = \frac{1+\xi}{1-\xi} = \frac{1-\bar{\xi}}{1+\bar{\xi}},$$

$$\xi = \frac{\int_0^l L\cos\frac{2\pi r}{l}dr}{\int_0^l Ldr}, \qquad \bar{\xi} = \frac{\int_0^l C\cos\frac{2\pi r}{l}dr}{\int_0^l Cdr}. \quad (55)$$

For the first resonance with both ends short-circuited, we have

$$\frac{4l^2}{\lambda^2} = \frac{1-\xi}{1+\xi} = \frac{1+\bar{\xi}}{1-\bar{\xi}}. \quad (56)$$

It will be observed from (39) and (42) that L is directly proportional and C inversely proportional to $K(r, \rho)$. Evidently, the approximations corresponding to χ and ξ on the one hand and to $\bar{\chi}$ and $\bar{\xi}$ on the other are not necessarily the same. In fact, in some extreme cases in which either L or C become infinite at one end too rapidly, one of the above formulas becomes entirely useless. For example, in a cylindrical cavity[35] for cylindrical waves C is directly proportional and L inversely proportional to the distance r from the axis.

[35] A section of a circular cylinder between two parallel planes.

For the principal resonance with voltage maximum at $r = 0$ and 0 voltage at $r = l$, we have

$$\frac{16l^2}{\lambda^2} = \frac{1+\chi}{1-\chi} = \frac{1-\bar{\chi}}{1+\bar{\chi}}, \qquad (57)$$

where χ and $\bar{\chi}$ are given by (54). The first formula is useless but the second gives

$$\frac{16l^2}{\lambda^2} = \frac{1+\dfrac{4}{\pi^2}}{1-\dfrac{4}{\pi^2}} = 2.35, \quad \frac{4l}{\lambda} = 1.53, \quad \frac{2\pi l}{\lambda} = 2.41. \quad (58)$$

The value of $2\pi l/\lambda$ as given by the exact theory should be the first root of $J_0(x) = 0$; this root is $2.40\cdots$.

In connection with antenna problems L becomes infinite in the approximate equations (39) but not rapidly enough to cause any trouble. Using these equations we obtain the following results for wires enclosed within fictitious "reflecting" spheres, with their centers at mid-points of the wires and with their surfaces passing through the ends of the wires.

For cylindrical wires

$$\chi = -\frac{\text{Si}\,\pi}{\pi\left(\log\dfrac{2l}{a} - 1\right)} = -\frac{120\,\text{Si}\,\pi}{\pi K_a} = -\frac{70.74}{K_a},$$

$$\xi = -\frac{\text{Si}\,2\pi}{2\pi\left(\log\dfrac{2l}{a} - 1\right)} = -\frac{27.08}{K_a}. \qquad (59)$$

For thin spheroidal wires

$$\chi = -\frac{\text{Si}\,\pi + 0.5\,\text{Si}\,2\pi}{\pi\log\dfrac{l}{a}} = -\frac{120(\text{Si}\,\pi + 0.5\,\text{Si}\,2\pi)}{\pi K_a} = -\frac{97.82}{K_a},$$

$$\xi = -\frac{\text{Si}\,2\pi - 0.5\,\text{Si}\,4\pi}{2\pi\log\dfrac{l}{a}} = -\frac{12.84}{K_a}. \qquad (60)$$

For wires with diamond-shaped longitudinal cross section

$$\chi = -\frac{\text{Si}\,\pi}{\pi\log\dfrac{2l}{a}} = -\frac{70.74}{K_a},$$

$$\xi = \frac{\text{Si}\,\pi - 0.5\,\text{Si}\,2\pi}{\pi\log\dfrac{2l}{a}} = \frac{43.66}{K_a}. \qquad (61)$$

Resonance in Very Thin Conical Antennas

We are now ready to consider the relations between lengths of antennas of various sizes and shapes and resonant wavelengths, under the condition that K is very large. Subsequently, we shall obtain approximate expressions for the input impedance of nonuniform antennas from which resonant lengths can be computed for antennas with moderate characteristic impedances.

In conical antennas L and C are distributed uniformly and the resonant wavelength is affected only by the end reactance. In so far as the first power of $1/K$ is concerned this effect is given by (30) which becomes

$$\frac{4l}{\lambda} = 1 - \frac{2F\left(\dfrac{\pi}{2}\right)}{\pi K} = 1 - \frac{120\,\text{Si}\,\pi + 60\,\text{Si}\,2\pi}{\pi K} = 1 - \frac{97.82}{K}$$

$$\frac{2l}{\lambda} = 1 - \frac{F(\pi)}{\pi K} = 1 - \frac{60\,\text{Si}\,2\pi - 30\,\text{Si}\,4\pi}{\pi K} = 1 - \frac{12.84}{K}, \qquad (62)$$

for the first and the second resonances, respectively.

Resonance in Very Thin Spheroidal Antennas

In spheroidal antennas we have to consider two effects: The end reactance and the nonuniform distribution of L and C. Ultimately, as K increases indefinitely, the two effects become additive. Thus for the first resonant wavelength, we have

$$\frac{4l}{\lambda} = 1 - \frac{2F\left(\dfrac{\pi}{2}\right)}{\pi K_a} - \chi = 1. \qquad (63)$$

Hence, in thin spheroidal antennas the deviation of the resonant length $2l$ from $\lambda/2$ is proportional at least to the square of $1/K$. This result agrees with Abraham's formula which, when expressed in terms of K_a, is

$$\frac{4l}{\lambda} = 1 - \frac{5040}{(K_a + 83)^2}. \qquad (64)$$

For the second resonance we have

$$\frac{2l}{\lambda} = 1 - \frac{F(\pi)}{\pi K_a} + \xi = 1 - \frac{2F(\pi)}{\pi K_a} = 1 - \frac{25.68}{K_a}. \qquad (65)$$

Resonance in Very Thin Cylindrical Antennas

Similarly at the first resonance in cylindrical antennas, we have

$$\frac{4l}{\lambda} = 1 - \frac{2F\left(\dfrac{\pi}{2}\right)}{\pi K_a} - \chi = 1 - \frac{60\,\text{Si}\,2\pi}{\pi K_a} = 1 - \frac{27.08}{K_a}. \qquad (66)$$

At the second resonance we shall have

$$\frac{2l}{\lambda} = 1 - \frac{F(\pi)}{\pi K_a} + \xi = 1 - \frac{120\,\text{Si}\,2\pi - 30\,\text{Si}\,4\pi}{\pi K_a} = 1 - \frac{39.92}{K_a}. \qquad (67)$$

Resonance in Very Thin Antennas with "Diamond-Shaped" Longitudinal Cross Sections

For these antennas we have

$$\frac{4l}{\lambda} = 1 - \frac{60\,\text{Si}\,2\pi}{\pi K_a} = 1 - \frac{27.08}{\pi K_a},$$

$$\frac{2l}{\lambda} = 1 + \frac{120 \, \mathrm{Si} \, \pi - 120 \, \mathrm{Si} \, 2\pi + 30 \, \mathrm{Si} \, 4\pi}{\pi K_a} = 1 + \frac{30.82}{K_a}, \quad (68)$$

for the first and second resonances, respectively.

Resonant Conditions Depend on the Shape of the Longitudinal Cross Section of the Antenna as Well as on the Size of the Transverse Cross Section

In its effect on the resonant length the average characteristic impedance K_a represents the "average size" of the cross section of the antenna. Since K_a depends on the logarithm of some mean radius, the resonant length varies rather slowly with the size of the cross section of the antenna.

The second factor is the *shape* of the longitudinal cross section. Page and Adams have supposed that a cylindrical wire is equivalent to a somewhat fatter and somewhat longer spheroid. On geometric grounds this appears reasonable; but our computations do not support the assumption. For a very thin cylindrical wire the "equivalent spheroid" would have to be very fat in comparison. In fact, if we let the radius of the cylinder approach zero, then the ratio of the base radius of the equivalent spheroid to the radius of the cylindrical wire will approach infinity.

The Input Impedance of Antennas with Variable Characteristic Impedance

We have already stated that in the first approximation a nonuniform line may be regarded as a uniform line with a characteristic impedance equal to the average characteristic impedance. The goodness of this approximation depends on the relative deviation of $K(r, \rho)$ from K_a. Even when these deviations are small for sections only a few wavelengths long, they will be prohibitively large for really long sections. For example, the average characteristic impedance of an infinitely long cylindrical wire is infinite; but the input

impedance of a cylindrical wire extending to infinity on both sides of a generator is certainly not infinite. As a matter of fact (see footnote 20), for thin wires this input impedance is approximately[36]

$$K(0) = 120 \log \frac{\lambda}{2a} - 207. \quad (69)$$

This impedance is represented by curve a in Fig. 21. Curve b is a plot of a more accurate equation (165). Fig. 22 is a plot of $K(0)$ over a wider range of radii.

[36] There is a reactance in shunt with the resistance (69); this reactance depends on the length of the segment over which the electromotive force is applied, and has little effect on the total impedance unless the segment is very short. Theoretically, the terminals of the generator could be brought so close together as to short-circuit it; but, in practice, this is not done.

In practice, however, antennas are never very long and we can treat them as transmission lines with slightly variable characteristic impedances. One solution for such lines, based on Picard's method of integrating differential equations, has been obtained by Carson.[6] Adapting his solution to our problem and retaining only the first order of correction terms, we obtain the following expression for the input impedance:

$$Z_i = K_a \frac{G \sin L + i[(F - N) \sin L - (K_a - M) \cos L]}{[(K_a + M) \sin L + (F + N) \cos L] - iG \cos L}; \quad (70)$$
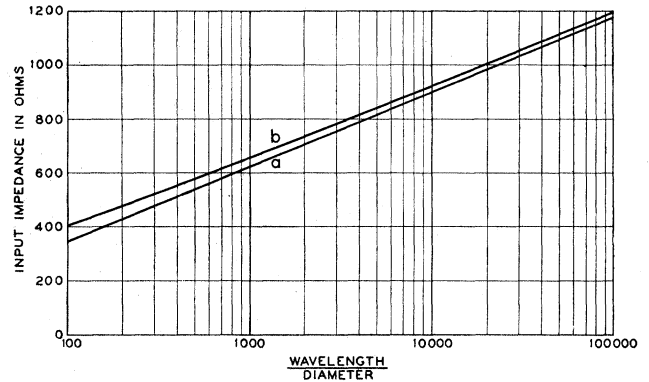

Fig. 21—The input impedance of an infinitely long cylindrical wire. Curve (a) is a plot of equation (69); curve (b) is a plot of the more accurate equation (165).

where M and N functions are defined by

$$M(L) = \beta \int_0^l [K_a - K(r, \rho)] \sin 2\beta r \, dr,$$

$$N(L) = \beta \int_0^l [K_a - K(r, \rho)] \cos 2\beta r \, dr. \quad (71)$$

The real and imaginary parts of the input impedance are

$$R_i = \frac{K_a G(K_a + N \sin 2L - M \cos 2L)}{G^2 \cos^2 L + [(K_a + M) \sin L + (F + N) \cos L]^2}$$

$$X_i = \frac{K_a[\tfrac{1}{2}(G^2 + F^2 + M^2 - N^2 - K_a^2) \sin 2L + (MN - K_a F) \cos 2L + (MF - K_a N)]}{G^2 \cos^2 L + [(K_a + M) \sin L + (F + N) \cos L]^2}. \quad (72)$$

For *cylindrical antennas* M and N functions become

$$M(L) = 60(\log 2L - \mathrm{Ci} \, 2L + C - 1 + \cos 2L),$$
$$N(L) = 60(\mathrm{Si} \, 2L - \sin 2L). \quad (73)$$

For antennas in free space, with a *rhombic* longitudinal cross section, or for vertical antennas of triangular shape[37] of base radius a, above a perfectly conducting ground, we have

$$M(L) = 60(C + \log 2L - \mathrm{Ci} \, 2L)(1 - \cos 2L)$$
$$\qquad - 60 \, \mathrm{Si} \, 2L \sin 2L,$$

$$N(L) = 60 \, \mathrm{Si} \, 2L - 60(C + \log 2L - \mathrm{Ci} \, 2L) \sin 2L,$$

$$K_a = 120 \log \frac{2l}{a}. \quad (74)$$

[37] Inverted conical antennas.

For spheroidal antennas we obtain

$$M(L) = 60(C + \log L - \text{Ci } 2L) + 30(\text{Si } 4L - 2 \text{ Si } 2L) \sin 2L$$
$$+ 30(C + \log 2L - 2 \text{ Ci } 2L + \text{Ci } 4L - 1) \cos 2L,$$

$$N(L) = 60 \text{ Si } 2L + 30(\text{Ci } 4L - \log L - C - 2 \log 2) \sin 2L$$
$$- 30 \text{ Si } 4L \cos 2L;$$

$$M(L) = G(L) - 60 \log 2 - 30(1 - \log 2) \cos 2L,$$

$$N(L) = F(L) - 60 \log 2 \sin 2L. \tag{75}$$

Diamond-shaped antennas can be treated more accurately if we take cognizance of the fact that the first
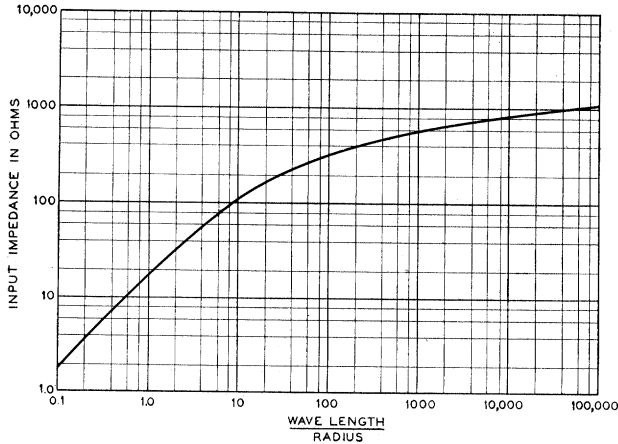

Fig. 22—The input impedance of the infinitely long cylindrical wire. For $0.1 < \lambda/2a \leqq 3$, this curve has been computed by numerical integration from (164); for $\lambda/2a \geqq 500$, the curve has been computed from the approximate equation (165); then, the two curves have been freely joined together.

half of the antenna has a uniform characteristic impedance and the second half, nonuniform. Thus we may obtain the input impedance into the second half from

as functions of l/λ for different values of K. In Fig. 25 we have the input resistance at resonance $(X_i = 0)$ in the vicinity of $l = \lambda/4$. Fig. 26 shows the resonant impedance in the vicinity of $l = \lambda/2$. The points (except the first) are values obtained experimentally by Feldman[39] for $\lambda = 9$ meters. The first point is the value obtained by Morrison and Smith[32] for the square tower.

Resonant lengths of cylindrical antennas are short of $\lambda/2$ and λ by the percentages shown in Fig. 27;
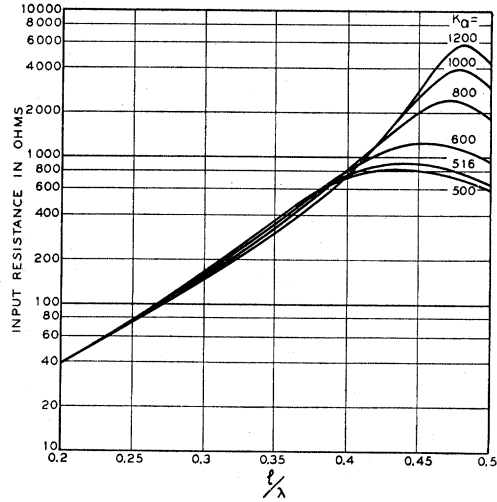

Fig. 23—The input resistance of cylindrical antennas in free space. For vertical antennas over a perfectly conducting ground divide the ordinates and K_a by 2.

curve 1 is for the principal resonance and curve 2 for the second resonance. The points above curve 1 have been taken from Englund,[16] those above curve 2 have been supplied by Feldman with one exception, the first point comes from the paper by Morrison and Smith.

$$Z_i = K_a \frac{G \sin \dfrac{L}{2} + i\left[(F - N) \sin \dfrac{L}{2} - (K_a - M) \cos \dfrac{L}{2} \right]}{\left[(K_a + M) \sin \dfrac{L}{2} + (F + N) \cos \dfrac{L}{2} \right] - iG \cos \dfrac{L}{2}} \tag{76}$$

and, then, compute the input impedance into the first half using (76) as its terminating impedance. The functions M and N to be used in (76) are[38]

$$M(L) = 60 \log 2 + 60(C + \log L - \text{Ci } 2L) \cos L$$
$$- 60 \text{ Si } 2L \sin L,$$

$$N(L) = 60(\text{Si } 2L - 2 \text{ Si } L) \cos L$$
$$- 60(C + \log L + \text{Ci } 2L - 2 \text{ Ci } L) \sin L,$$

$$K_a = 120 \log \frac{2l}{a} \cdot \tag{77}$$

Input Impedance Curves for Cylindrical Antennas

In Figs. 23 and 24 the input resistance and the reactance of cylindrical antennas in free space are shown

In this latter case, it is known that the base insulator is responsible for about 30 micromicrofarads in shunt with the antenna; this capacitance is more than sufficient to account for the difference of $2\frac{1}{2}$ per cent between the measured and computed values. On the other hand, the tower is tapered near the base and consequently has less capacitance than it would have had if it were not tapered; we feel confident that if all these factors are taken into consideration the discrepancy between the theory and experiment would be quite

[38] In (76) G, F, M, and N are functions of L, as indicated by the first term in the numerator, while the argument of the sines and cosines is $L/2$.

[39] Feldman's measurements were made for vertical wires of different sizes; we have doubled his results to obtain the values corresponding to the free-space condition.

small. Hence, it is unnecessary to postulate the existence of a lumped series inductance of 6.8 microhenrys and a lumped shunt capacitance of 200 micromicrofarads at the base of the antenna in order to explain the measurements. These values were assumed, but not accounted for, by Morrison and Smith in order to bring their measurements into agreement with formulas from Siegel and Labus.

We now believe that the antenna theory is in such a state that accurate results can be calculated if all "visible" factors, such as base capacitances and antenna shapes are taken into consideration. The imperfect conductivity of the ground does not appear to affect the results.
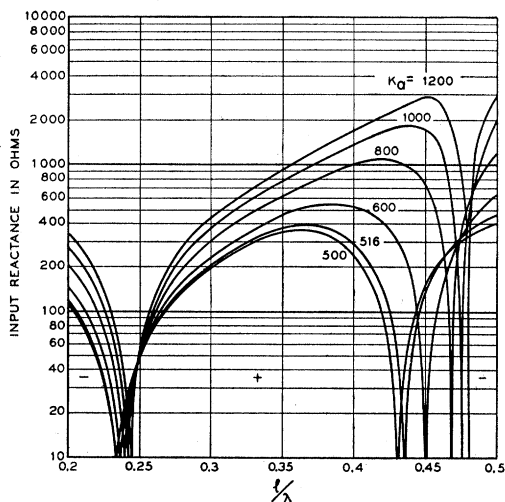

Fig. 24—The input reactance of cylindrical antennas in free space. For vertical antennas over a perfectly conducting ground divide the ordinates K_a by 2.

For some unexplained reason the impedance curves for cylindrical antennas published by King are considerably in error in some regions. Thus, when $\log 2l/a = 10.58$ so that $K_a = 1150$, King obtains[40] $R_{max} = 16,000$ and $X_{max} = 9000$; our values are $R_{max} = 5500$ and $X_{max} = 3000$. Available measurements of R_{max} support our results (Fig. 26).

The Input Impedance for $l = \lambda/4$ and $l = \lambda/2$

When $l = \lambda/4$ and $= \lambda/2$, we have, respectively,

$$Z_i = \frac{K_a}{K_a + M\left(\frac{\pi}{2}\right)}\left[G\left(\frac{\pi}{2}\right) + iF\left(\frac{\pi}{2}\right) - iN\left(\frac{\pi}{2}\right)\right]$$

$$Z_i = \frac{K_a[K_a - M(\pi)]}{G^2 + (F+N)^2}\left[G(\pi) - iF(\pi) - iN(\pi)\right]. \qquad (78)$$

The quantity $G(\pi/2) = 73.13$ is the input resistance of a half-wave antenna for $K_a = \infty$, that is, for an infinitely thin wire. For any finite value of K_a, the input resistance depends also on $M(\pi/2)$, that is, on the shape of the longitudinal cross section of the antenna.

[40] King's values have been doubled to obtain the free-space figures.

For example, for cylindrical and spheroidal antennas $M(\pi/2)$ is equal, respectively, to -21 and 41. Hence, when $l = \lambda/4$, the input resistance of cylindrical antennas is somewhat higher than 73.13 and the input resistance of spheroidal antennas is somewhat lower than 73.13.
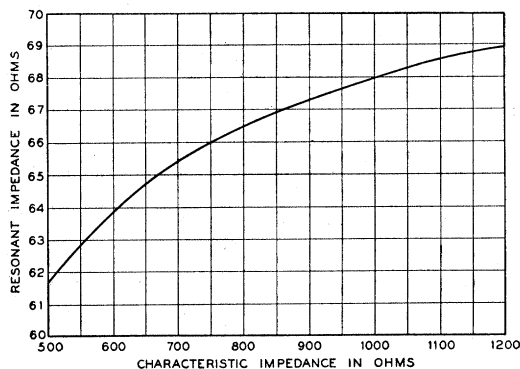

Fig. 25—The resonant impedance of cylindrical antennas as a function of K, when l is in the vicinity of $\lambda/4$.

When $l = \lambda/4$ and $K_a = \infty$, the input reactance of cylindrical antennas is $30\,\text{Si}\,2\pi = 42.5$ and that of spheroidal antennas is 0. For cylindrical antennas with finite characteristic impedances the input reactance is somewhat higher than 42.5 and for spheroidal antennas it is still zero. These results do not include terms depending on $1/K_a$. From Abraham's expression for the resonant frequencies and from (70), we find that the approximate value of the input reactance of thin spheroidal antennas, when $l = \lambda/4$, is $7900(K_a - 41)/(K_a + 83)^2$.

Bent Antennas

Our method of antenna analysis is applicable to bent antennas (Fig. 28). From the theory of principal waves guided by thin diverging conical wires[21] we observe
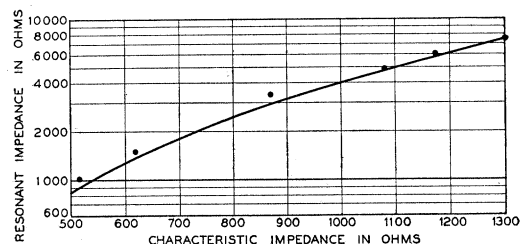

Fig. 26—The resonant impedance of cylindrical antennas is a function of K, when l is in the vicinity of $\lambda/2$.

that the capacitance between two elements at A and A' depends only on the distance between them and on their radii. For equal radii, we shall have then

$$K(x, a) = 120 \log \frac{d(x)}{a}. \qquad (79)$$

For unequal radii, a is the geometric mean of the actual radii.

From (79) we can obtain the average characteristic impedance; then from (71) we compute M and N functions which are needed to correct transmission equations

of the principal waves for the nonuniform distribution of the inductance and capacitance. The function $G(L)$ is obtained from (25): First we compute the radiated power on the assumption that $a = 0$ and that the current distribution is sinusoidal with current nodes at the open ends of the antenna; then, we divide this power by $\frac{1}{2}$ of the square of the current amplitude. This calculation can be carried out by either of two methods: (1) The Poynting flux method and (2) the produced-electromotive-force method. Only the second of these methods can be used for calculating $F(L)$; moreover, in this calculation a must be assumed not
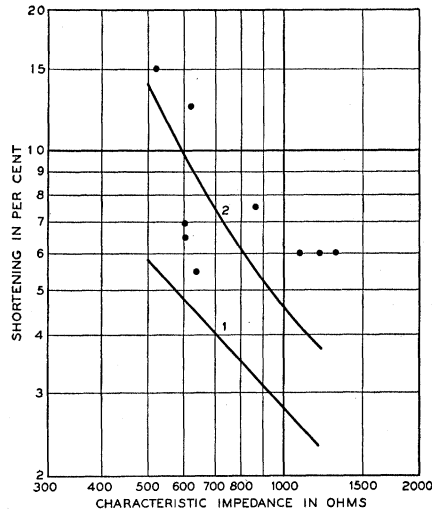

Fig. 27—The resonant length of cylindrical antennas in free space. Curve 1 is for the principal resonance. Curve 2 is for the second resonance. The ordinates represent the required shortening in per cent.

merely infinitely small but also varying with x in such a way that $d(x)/a$ remains constant.

When these functions have been computed, the input impedance is obtained from (70).

Loop Antennas

If an antenna is bent into a complete loop and the ends are joined together, the procedure for calculating G and F functions is the same as that outlined above except that the sinusoidal current distribution must be chosen to have an antinode instead of a node at the far end of the antenna. Naturally, M and N functions are not affected by the conditions at the far end.

Current Distribution

In order to find the current distribution in nonuniform antennas we should calculate the principal current distribution from Carson's equations,[6] assuming that the transmission line is terminated by the output radiation impedance as given in this paper, and superimpose on it the secondary current distribution. The latter can be obtained from (11) and (36).

Knowledge of current distribution is needed largely in computing radiation patterns, and these are affected by the ground conditions to a much greater extent than by the changes in the current distribution. It is

doubtful, therefore, that the labor expended in calculating current distributions, light as it is, would be commensurate with the practical value of the results. However, the current distribution can be measured and, hence, used for checking the theory.

V. Antennas with Uniform Characteristic Impedances: Mathematical Analysis

A set of equations for transverse magnetic spherical waves and its general discussion may be found in previous papers.[21,41] This set will now be developed to suit the needs of our particular problem. In the first place we shall assume that $\partial/\partial\phi = 0$. It may appear that this assumption will restrict the analysis to coaxial cones; in reality, however, the solutions of the restricted equations can be used for the more general case of two conical wires inclined to each other.

Thus, if $\partial/\partial\phi = 0$ our fundamental equations become

$$rE_\theta = -\frac{\partial V}{\partial\theta}, \qquad i\omega\epsilon r^2 E_r = n(n+1)A,$$

$$rH_\phi = -\frac{\partial A}{\partial\theta}, \qquad V = -\frac{1}{i\omega\epsilon}\frac{\partial A}{\partial r}. \qquad (80)$$

where n is a constant depending upon the boundary conditions at the surface of the antenna and the "flux function" A is

$$A^+ = \sqrt{\beta r}\left[J_{n+1/2}(\beta r) - iN_{n+1/2}(\beta r)\right]T(\theta),$$

$$A^- = \sqrt{\beta r}\,J_{n+1/2}(\beta r)T(\theta). \qquad (81)$$

The function A^+ defines the outward-bound progressive wave while A^- the standing wave finite at the

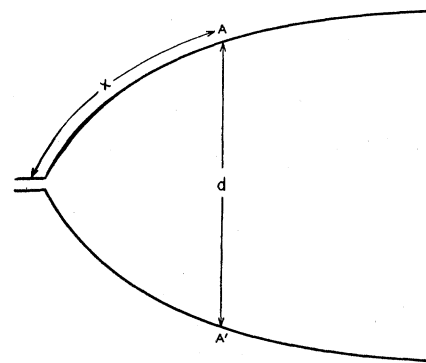Fig. 28—A bent antenna.

origin $r = 0$. The function $V(r, \theta)$ is the "potential" defined as the line integral of E_θ along any path beginning on some fixed radius $\theta = \theta'$ and contained entirely in the co-ordinate sphere of radius r. The function $T(\theta)$ is a solution of Legendre's equation

$$T(\theta) = pP_n(-\cos\theta) + qP_n(\cos\theta), \quad \text{if} \quad n \neq 0,$$

$$= p\log\cot\frac{\theta}{2}, \quad \text{if} \quad n = 0. \qquad (82)$$

[41] W. L. Barrow, L. W. Chu, and J. J. Jansen, "Biconical electromagnetic horns," Proc. I.R.E., vol. 27, pp. 769–779; December, 1939.

The function $P_n(x)$ is so defined that

$$P_n(1) = 1. \tag{83}$$

If n is not an integer, $P_n(x)$ has a logarithmic singularity at $x = -1$. If n is an integer, $P_n(-x) = (-)^n P_n(x)$ and (82) is not the most general solution of Legendre's equation. There is a second solution $Q_n(x)$ which is singular at two points $x = \pm 1$.

The boundary conditions are: the electric intensity E_r must vanish at the surface of the antenna and the field must be finite. If the conical conductors are of finite length, the entire space must be divided by coordinate spheres into regions: (1) free from conductors, (2) containing only one conductor, and (3) containing two conductors.

In the region free from conductors n must necessarily be an integer because for nonintegral values of n Legendre's equation possesses no solutions finite for all values of θ. The function $Q_n(\cos \theta)$ being singular at $\theta = 0, \pi$ is also out of the picture. Thus in such regions the complete solution is a series of functions $P_n(\cos \theta)$ corresponding to different integral values of n.

In the region containing only one cone with its axis along $\theta = 0$, the coefficient q in (82) must vanish because $P_n(\cos \theta)$ is infinite where $\theta = \pi$. Thus (82) becomes

$$T(\theta) = p P_n(-\cos \theta). \tag{84}$$

The singularity of this function at $\theta = 0$ is, of course, excluded by the conductor. If ψ is the angle made by the generators of the conical conductor with the axis, then E_r will vanish on the conductor if

$$P_n(-\cos \psi) = 0. \tag{85}$$

This is the equation for the constant n.

In the region containing two conductors, one with its axis along $\theta = 0$ and the other along $\theta = \pi$, we must have $T(\psi_1) = T(\pi - \psi_2) = 0$ and therefore

$$p P_n(-\cos \psi_1) + q P_n(\cos \psi_1) = 0,$$
$$p P_n(\cos \psi_2) + q P_n(-\cos \psi_2) = 0. \tag{86}$$

Eliminating p and q, we have the equation for n,

$$\frac{P_n(-\cos \psi_1)}{P_n(\cos \psi_2)} = \frac{P_n(\cos \psi_1)}{P_n(-\cos \psi_2)}. \tag{87}$$

If the cone angles ψ_1 and ψ_2 are small their cosines are nearly equal to unity, $P_n(\cos \psi_1) \doteq P_n(\cos \psi_2) \doteq 1$ and (87) becomes approximately

$$P_n(-\cos \psi_1) P_n(-\cos \psi_2) = 1. \tag{88}$$

If the cones are equal $\psi_1 = \psi_2 = \psi$ then

$$P_n(-\cos \psi) = \pm P_n(\cos \psi), \quad \text{and} \quad p = \mp q. \tag{89}$$

For small values of ψ, this becomes

$$P_n(-\cos \psi) = \pm 1. \tag{90}$$

If the two cones are inclined toward each other and if their angles ψ_1 and ψ_2 are so small that the "proximity effect" is negligible, we can express the total field as the resultant of two fields, associated with the individual cones, and the function that must vanish on the boundaries becomes

$$T(\theta_1, \theta_2) = p P_n(-\cos \theta_1) + q P_n(-\cos \theta_2), \tag{91}$$

where θ_1 and θ_2 are the angles made by typical radius with the axes of the conductors. If the angle between the axes is ζ, we have approximately

$$p P_n(-\cos \psi_1) + q P_n(-\cos \zeta) = 0,$$
$$p P_n(-\cos \zeta) + q P_n(-\cos \psi_2) = 0. \tag{92}$$

The equation for n will then become

$$P_n(-\cos \psi_1) P_n(-\cos \psi_2) = [P_n(-\cos \zeta)]^2. \tag{93}$$

For equal cones, we have

$$P_n(-\cos \psi) = \pm P_n(-\cos \zeta), \quad p = \mp q. \tag{94}$$

Inasmuch as with reference to a single co-ordinate system $T(\theta_1, \theta_2)$ is a function of two spherical co-ordinates θ and ϕ the field, if wanted, must be computed from the general equations[21] and not from (80).

The boundary condition $E_r = 0$ on the surface of the antenna will also be satisfied (see (80)) if E_r vanishes *everywhere;* this happens when $n = 0$. In this case the nontrivial solution is singular for two values of θ so that the wave can exist only in the presence of two conductors excluding the corresponding radii from the field. This is the so-called "principal" wave and its properties have already been discussed in detail.[21]

In the region containing two conductors, the transverse voltage $V_n(r)$ between the conductors, corresponding to any transmission mode except the principal, vanishes,

$$V(r) = 0. \tag{95}$$

This is true because $V(r)$ is the difference between the values of $V(r, \theta)$ on the two conductors and these values, being proportional to $T(\theta)$, vanish with $T(\theta)$; and it is only for the principal wave that $T(\theta)$ does not vanish on the conductors. Hence, the total transverse voltage $V(r)$ between two conductors depends entirely on the principal wave.

Let $I'_n(r)$ be the electric current in the upper cone of the double cone and $I''_n(r)$ the corresponding current in the lower cone (Fig. 5). Assuming that the upward direction is positive, we have

$$I_n'(r) = 2\pi r \sin \psi H_\phi(r, \psi) = -2\pi \sin \psi \frac{\partial A}{\partial \psi},$$

$$I_n''(r) = 2\pi r \sin \psi H_\phi(r, \pi - \psi) = 2\pi \sin \psi \frac{\partial A}{\partial \theta}\bigg|_{\theta = \pi - \psi}. \tag{96}$$

For all values of $n > 0$ the function $\sqrt{x} N_{n+\frac{1}{2}}(x)$ is

infinite at $x=0$; therefore, the proper expression for A is A^- (see (81)) and

$$I_n'(r) = -2\pi\sqrt{\beta r}\, J_{n+1/2}(\beta r)\sin\psi\frac{\partial}{\partial\psi}T(\psi),$$

$$I_n''(r) = 2\pi\sqrt{\beta r}\, J_{n+1/2}(\beta r)\sin\psi\frac{\partial}{\partial\psi}T(\pi-\psi). \quad (97)$$

Since for $n>0$, the function $\sqrt{x}J_{n+\frac{1}{2}}(x)$ vanishes at $x=0$, we have

$$I_n'(0) = I_n''(0) = 0. \quad (98)$$

At the apex the electric current is determined solely by the principal wave.

Thus, we have justified equations (10) and the equivalent circuit for double cones of equal length that follows from these equations. It was shown there that the first approximation for the output impedance (19) can be obtained in the form (26) by a method quite different from the present and yielding the results in their simplest form. On the other hand, the present method permits the use of successive approximations and is quite general. For this reason, we are justified in developing it still further.

In the case of a double cone (Fig. 5) different transmission modes are determined by the roots of (89) or, for small cone angles ψ, by (90). The latter equation is a very good approximation even if ψ is as large as 0.1 since $\cos\psi = 1 - \frac{1}{2}\psi^2$ will differ from unity by less than 1 per cent and $P_n(1) = 1$. From the theory of Legendre's functions, we have

$$P_n(-\cos\theta) = \frac{\sin n\pi}{\pi}\sum_{\alpha=0}^{\infty}\frac{(-)^\alpha\Gamma(n+\alpha+1)}{\Gamma(n-\alpha+1)}\Bigg[2\log\sin\frac{\theta}{2}$$

$$+\psi(n+\alpha)+\psi(-n+\alpha-1)-2\psi(\alpha)\Bigg]\frac{\sin^{2\alpha}\frac{\theta}{2}}{\alpha!\alpha!}, \quad (99)$$

where

$$\psi(x) = \frac{\Gamma'(x+1)}{\Gamma(x+1)},\; \psi(-x-1) = \psi(x) + \pi\cot\pi x. \quad (100)$$

For small values of ψ the term corresponding to $\alpha=0$ is dominant and we have approximately

$$P_n(-\cos\theta) = \frac{2}{\pi}\sin n\pi\Bigg[\log\frac{\theta}{2} + \psi(n) - \psi(0)\Bigg]$$

$$+ \cos n\pi. \quad (101)$$

Hence, the equation

$$P_n(-\cos\psi) = 1 \quad (102)$$

becomes

$$\tan\frac{n\pi}{2} = -\frac{2}{\pi}\Bigg[\log\frac{2}{\psi} + \psi(0) - \psi(n)\Bigg]$$

$$= -\frac{2}{\pi}\Bigg[\frac{K}{120} + \psi(0) - \psi(n)\Bigg], \quad (103)$$

where K is the characteristic impedance to the principal waves.[21]

Similarly, the equation

$$P_n(-\cos\psi) = -1 \quad (104)$$

becomes

$$\cos\frac{n\pi}{2} = \frac{2}{\pi}\Bigg[\frac{K}{120} + \psi(0) - \psi(n)\Bigg]. \quad (105)$$

As the characteristic impedance K increases indefinitely, the roots of (103) approach

$$n = 2m + 1 + \frac{120}{K}, \quad m = 0, 1, 2, \cdots. \quad (106)$$

Likewise, the roots of (105) approach

$$n = 2m + \frac{120}{K}, \quad m = 0, 1, 2, \cdots. \quad (107)$$

The roots of the characteristic equation (85) for a single cone approach

$$n = m + \frac{60}{K}, \quad m = 0, 1, 2, \cdots. \quad (108)$$

When n is a root of (102) then by virtue of (89) the corresponding field function $T(\theta)$ is proportional to

$$L_{2m+1}(\theta) = \tfrac{1}{2}\lfloor P_{2m+1+\Delta}(\cos\theta) - P_{2m+1+\Delta}(-\cos\theta)\rfloor, \quad (109)$$

where

$$\Delta = \frac{120}{K}. \quad (110)$$

The derivatives of the L functions at $\theta=\psi$ and $\theta=\pi-\psi$ are approximately

$$\frac{dL(\psi)}{d\psi} = -\frac{dL(\pi-\psi)}{d\psi} = \frac{\sin\pi\Delta}{\pi\psi} = \frac{\Delta}{\psi} = \frac{120}{K\psi}. \quad (111)$$

Hence, the electric currents in the antenna, associated with these waves are nearly inversely proportional to the characteristic impedance and they flow in the same direction at points equidistant from 0.

When n is a root of (104), the field function $T(\theta)$ is proportional to

$$L_{2m}(\theta) = \tfrac{1}{2}\big[P_{2m+\Delta}(\cos\theta) + P_{2m+\Delta}(-\cos\theta)\big]. \quad (112)$$

In this case

$$\frac{dL_{2m}(\psi)}{d\psi} = \frac{dL(\pi-\psi)}{d\psi} = \frac{\sin\pi\Delta}{\pi\psi} = \frac{120}{K\psi}, \quad (113)$$

and the electric currents at points equidistant from 0 are in opposite directions

For small values of Δ, we have approximately

$$P_{n+\Delta}(\cos\theta) = P_n(\cos\theta)$$

$$+ 2\Delta\Bigg[P_n(\cos\theta)\log\cos\frac{\theta}{2} + S_n'\Bigg]$$

$$P_{n+\Delta}(-\cos\theta) = (-)^n P_n(\cos\theta)$$
$$+ 2\Delta\left[(-)^n P_n(\cos\theta)\log\sin\frac{\theta}{2} + S_n''\right],$$

$$S_n' = \sum_{\alpha=1}^n \frac{(-)^\alpha(n+\alpha)!}{\alpha!\alpha!(n-\alpha)!}\left(\frac{1}{n+\alpha} + \frac{1}{n+\alpha-1} + \cdots\right.$$
$$\left. + \frac{1}{n+1}\right)\sin^{2\alpha}\frac{\theta}{2},$$

$$S_n'' = \sum_{\alpha=1}^n \frac{(-)^\alpha(n+\alpha)!}{\alpha!\alpha!(n-\alpha)!}\left(\frac{1}{n+\alpha} + \frac{1}{n+\alpha-1} + \cdots\right.$$
$$\left. + \frac{1}{n+1}\right)\cos^{2\alpha}\frac{\theta}{2}. \tag{114}$$

Also, $\Delta \to 0$, we have

$$L_n(\theta) \to P_n(\cos\theta). \tag{115}$$

Thus, the characteristic functions for the region containing two cones approach the characteristic functions for free space except that free space cannot support the principal wave.

This property may be used for obtaining the second approximation to the solution of the antenna problem. We shall write general solutions appropriate to the free space, and to the "antenna region" within the sphere of radius l (Fig. 5). The two solutions must satisfy certain continuity conditions at the boundary sphere and it is in this matching operation that (115) is useful.

In the antenna region we can write

$$2\pi i\omega\epsilon r^2 E_r = \sum_n a_n \frac{\sqrt{\beta r}\,J_{n+1/2}(\beta r)}{\sqrt{\beta l}\,J_{n+1/2}(\beta l)}L_n(\theta),$$

$$\bar{I}(r) = 2\pi r\sin\psi H_\phi$$
$$= -\sum_n \frac{a_n\sqrt{\beta r}\,J_{n+1/2}(\beta r)}{n(n+1)\sqrt{\beta l}\,J_{n+1/2}(\beta l)}\sin\psi\,\frac{d}{d\psi}L_n(\psi), \tag{116}$$

where $\bar{I}(r)$ is the total current in the upper cone, associated with all waves except the principal. We shall call this current *complementary*. The summation in (116) is extended over the roots of (89), excepting $n = 0$.

In the free space we write

$$2\pi i\omega\epsilon r^2 E_r$$
$$= \sum_{k=1}^{\infty} b_k \frac{\sqrt{\beta r}\,[J_{k+1/2}(\beta r) - iN_{k+1/2}(\beta r)]}{\sqrt{\beta l}\,[J_{k+1/2}(\beta l) - iN_{k+1/2}(\beta l)]}P_k(\cos\theta). \tag{117}$$

At $r = l$, C must be continuous

$$\sum_n a_n L_n(\theta) = \sum_{k=1}^{\infty} b_k P_k(\cos\theta). \tag{118}$$

But as $K \to \infty$ and $\psi \to 0$, we have (115) and, therefore, in the limit the coefficients a_n and b_k must be correspondingly equal.

The coefficients b_k can be easily determined for the

limiting case $K = \infty$. In the first place $\bar{I}(r) \to 0$ as $K \to \infty$. This can be shown by substituting from (111) into (116) and obtaining

$$\bar{I}(r) = -\frac{120}{K}\sum_n \frac{a_n\sqrt{\beta r}\,J_{n+1/2}(\beta r)}{n(n+1)\sqrt{\beta l}\,J_{n+1/2}(\beta l)}. \tag{119}$$

The only term which would not approach zero is the term corresponding to $m = 0$ in (107). This term is entirely absent, however, in the symmetric case of Fig. 5. In the case of cones of unequal length this term would be present and we would exclude it from $\bar{I}(r)$ and consider it separately as the *"principal antisymmetric current wave."* In either case, the conclusion is that as $K \to \infty$, *the current distribution in the antenna tends to become strictly sinusoidal.*

If the current distribution is known, the field can be determined by the retarded-potential method. For a sinusoidal distribution of amplitude I_0, with current nodes at the ends of the antenna, we have

$$C = 2\pi r\sin\theta H_\phi$$
$$= -iI_0(e^{-i\beta r}\cos\beta l - \tfrac{1}{2}e^{-i\beta r_1} - \tfrac{1}{2}e^{-i\beta r_2}), \tag{120}$$

where r_1 and r_2 are distances from the upper and the lower ends of the antenna. At great distances from the antenna this becomes

$$C = iI_0[\cos(\beta l\cos\theta) - \cos\beta l]e^{-i\beta r}. \tag{121}$$

From Maxwell's equations we have

$$2\pi i\omega\epsilon r^2 E_r = \frac{1}{\sin\theta}\frac{\partial C}{\partial\theta} = i\beta l\sin(\beta l\cos\theta)e^{-i\beta r}. \tag{122}$$

On the other hand, at great distances from the antenna (117) becomes

$$2\pi i\omega\epsilon r^2 E_r = \sum_{k=1}^{\infty}\sqrt{\frac{2}{\pi}}\,i^{k+1}b_k(\beta l)^{-1/2}[J_{k+1/2}(\beta l)$$
$$- iN_{k+1/2}(\beta l)]^{-1}P_k(\cos\theta)e^{-i\beta r}. \tag{123}$$

Hence, in order to find b_k we only have to expand $\sin(\beta l\cos\theta)$ into a series of Legendre's functions. This expansion is known to be

$$\sin(\beta l\cos\theta)$$
$$= \sum_{m=0}^{\infty}(-)^m(4m+3)\sqrt{\frac{\pi}{2\beta l}}\,J_{2m+3/2}(\beta l)P_{2m+1}(\cos\theta). \tag{124}$$

Substituting in (122) and comparing the result with (123) we obtain

$$b_{2m+1} = -i(2M+\tfrac{3}{2})\pi L J_{2m+3/2}(L)[J_{2m+3/2}(L) - iN_{2m+3/2}(L)]$$
$$b_{2m} = 0, \qquad L = \beta l = \frac{2\pi l}{\lambda}. \tag{125}$$

Letting in (119) $a_n = b_{2m+1}$, we obtain the asymptotic value for the complementary current as given by (16) and (18).

The above matching of radial electric fields in the

two regions can be regarded as the second step in a sequence of successive approximations. If we were to compute E_θ, we would discover that it is discontinuous at the boundary sphere $r = l$. By adding a proper expression to the external field, we can make E_θ continuous again and after matching E_r for the second time we shall have a more accurate expression for the complementary current. In all these higher approximations we no longer can make the coefficients a_n and b_k equal but have to expand the functions to be matched in series of Legendre's functions, $L_n(\theta)$ or $P_n(\cos\theta)$ as the case may be.

These functions are orthogonal[42] and, therefore, the coefficients of expansions are expressed by definite integrals.

VI. THE "INDUCED-ELECTROMOTIVE-FORCE" METHOD OF COMPUTING RADIATION AND THE RADIATION PARADOX

The "induced-electromotive-force" method[43] of computing radiation consists in obtaining the work done (and thus the energy contributed to the field) in driving given electric currents against the electric forces produced by them. This method furnishes not only the power lost by the source in radiation but also the average reactive power, that is, the average interchange of energy between the field and the generator.

For a linear current filament situated along the z axis and extended from $z = -l$ to $z = l$, the complex power Ψ is

$$\Psi = -\frac{1}{2}\int_{-l}^{l} E_z(z)I^*(z)dz, \qquad (126)$$

where E_z is the electric intensity parallel to the filament at the surface of the latter. If the filament consists of two perfectly conducting wires, one extending from $z = -l$ to $z = -s$ and the other from $z = s$ to $z = l$, and if the "generator" between $z = -s$ and $z = s$ consists of some means for transferring an electric charge from one wire to the other, then (126) becomes

$$\Psi = -\frac{1}{2}\int_{-s}^{s} E_z(z)I^*(z)dz \qquad (127)$$

since $E_z = 0$ on the surface of a perfect conductor.

Furthermore, if we assume that in the short interval $(-s, s)$ occupied by the source of energy the current $I(z)$ is equal to a constant I, (127) becomes

$$\Psi = -\frac{1}{2}I^*\int_{-s}^{s} E_z(z)dz. \qquad (128)$$

Since "action equals reaction," we have

$$V + \int_{-s}^{s} E_z(z)dz = 0, \qquad (129)$$

---

[42] It should be noted, however, that $L_0(\theta)$ is not a member of the orthogonal set $[L_n(\theta)]$ but $dL_0(\theta)/d(\theta)$ *is* a member of the orthogonal set $[dL_n(\theta)/d\theta]$.

[43] For a brief history of this method see S. A. Schelkunoff, "A general radiation formula," PROC. I.R.E., vol. 27, pp. 660–666; October, 1939.

where V is the "applied" electromotive force, that is, the force transferring the charge from one wire to the other, while the integral is the counterelectromotive force of the field. Hence, (128) may be rewritten as

$$\Psi = \tfrac{1}{2}VI^*. \qquad (130)$$

In order to find Ψ, it is necessary to compute the current I flowing through the generator in response to the electromotive force V—a major problem in itself.

The approximate method of computing Ψ is based on the following facts: (1) In thin wires the current $I(z)$ is nearly sinusoidal, (2) the field and, in particular, $E_z(z)$ can be calculated from $I(z)$ by the retarded-potential method, and (3) the error in the field becomes smaller as the error in $I(z)$ becomes smaller. This method has already been applied successfully to find the approximate power radiated from thin cylindrical antennas.

In the case of conical antennas we have

$$\Psi = -\frac{1}{2}\int_{-l}^{l} E_r(r)I^*(r)dr, \qquad (131)$$

for the average work done in order to maintain electric current in the lateral conical surface. To this we must add another integral if the top and bottom surfaces of the cone are large enough to make a noticeable contribution to Ψ. It is from this integral that we have computed $F(L)$ as given by (26). For thin antennas $G(L)$ is independent of the shape of the antennas and can be obtained equally well from (126). In order to carry out the necessary computations, the following expressions for the field produced by an infinitely thin filament supporting sinusoidal current have been used

$$I(z) = I\sin\beta(l - z), \qquad z > 0;$$
$$= I\sin\beta(l + z), \qquad z < 0;$$

$$E_z = 30iI\left(2\frac{e^{-i\beta r}}{r}\cos\beta l - \frac{e^{-i\beta r_1}}{r_1} - \frac{e^{-i\beta r_2}}{r_2}\right),$$

$$\rho E_\rho = 30iI\left[\frac{(z - l)e^{-i\beta r_1}}{r_1} + \frac{(z + l)e^{-i\beta r_2}}{r_2}\right.$$
$$\left. -\frac{2ze^{-i\beta r}}{r}\cos\beta l\right]$$

$$2\pi\rho H_\phi = -iI(e^{-i\beta r}\cos\beta l - \tfrac{1}{2}e^{-i\beta r_1} - \tfrac{1}{2}e^{-i\beta r_2}). \qquad (132)$$

In these equations r_1, r, and r_2 are, respectively, the distances from the lower end, the center, and the upper end of the antenna; ρ is the distance from the z axis.

In view of (23) and (25), $G(L)$ and $F(L)$ can be obtained correctly in this manner.

In (126) the quantity $-E_z(z)$ is the applied electromotive force per unit length and, therefore, $-E_z(z)/I(z)$ is the radiation impedance per unit length. If the antenna is perfectly conducting, this impedance is 0 along the entire antenna except in the region from $z = -s$ to $z = s$, occupied by the source of energy. The

conclusion is not startling, since after all energy does come out of its source.

What we have just said does not vitiate our previous assertion to the effect that an approximate value of Ψ can be obtained from an approximate value of $I(z)$. It is true that if $I(z)$ is taken to be a sinusoidal function of z, then E_z is different from zero and it would appear that there exists a nonvanishing distributed radiation impedance; but this means merely that the power given out by the "point generator" is approximately equal to the power given out by a certain continuous distribution of generators along the entire antenna. If the sinusoidal function for $I(z)$ is replaced by a more accurate function, then E_z will become smaller along the greatest part of the antenna but it will become larger in the interval from $z = -s$ to $z = s$. What the discussion in connection with (7) tells us is that if K is large, the more accurate expression for Ψ will not differ very much from the one obtained on the basis of sinusoidal distribution of current.

On the other hand, it is perfectly obvious that in the case of an antenna fed from a point generator the distributed radiation resistance $-E_z(z)/I(z)$, obtained from sinusoidal distribution, should not possess any significance since it is bound to approach zero as $I(z)$ is made to approach its exact value.

Feldman has communicated to the author that by taking the real part of $-E_z(z)/I(z)$, computed on the assumption that $I(z)$ is sinuscidal, and regarding this real part as a distributed series resistance added by radiation to the ohmic resistance, he computed the current and charge distributions along a long wire. Then he measured the shape of the minima of the current and of the charge. The measured values agreed well with the computed values.

Now, there is no question that there exists a distribution of series impedance that will represent correctly the effect of radiation on the *total current* and the *total charge* in the antenna (Section VII). What is surprising is that this resistance has been obtained fairly accurately as the real part of $-E_z(z)/I(z)$. After all, if this were a correct method of finding the equivalent distributed resistance, we should expect that its exact value will be found when the exact value of $I(z)$ is used. And yet if the exact value of $I(z)$ is used, then $-E_z(z)/I(z)$ must be equal to zero because this is precisely the boundary condition from which the exact current must be obtained!

VII. CURRENT-CHARGE EQUATIONS

The customary form of transmission equations for a line with distributed series resistance R, series inductance L, and shunt capacitance C is

$$\frac{dV}{dx} = -(R + i\omega L)I, \qquad \frac{dI}{dx} = -i\omega CV. \quad (133)$$

Since the electric charge per unit length is $q = CV$, the above equations can be written also in the current-

charge form

$$\frac{dq}{dx} = -(RC + i\omega LC)I, \qquad \frac{dI}{dx} = -i\omega q. \quad (134)$$

provided, however, that C is independent of x. The second equation of this set expresses the law of conservation of electric charge.

For a conical antenna (for any antenna for that matter) the second equation is satisfied automatically

$$\frac{dI}{dr} = -i\omega q. \quad (135)$$

Differentiating this with respect to r, we have

$$\frac{dq}{dr} = \frac{i}{\omega} \frac{d^2I}{dr^2}. \quad (136)$$

It has been shown in Section V that the total current $I(r)$ is the sum of the principal current $I_0(r)$ and an infinite number of complementary currents corresponding to characteristic values of n

$$I(r) = I_0(r) + \sum_n I_n(r). \quad (137)$$

Each component of the current satisfies the following equation[21]

$$\frac{d^2I_n}{dr^2} + \left[\beta^2 - \frac{n(n+1)}{r^2}\right]I_n = 0, \quad \beta = \frac{2\pi}{\lambda} = \frac{\omega}{v}. \quad (138)$$

Substituting the second derivative from this equation into (136), we obtain

$$\frac{dq}{dr} = -\frac{i\omega}{v^2} I(r) + \frac{i\omega}{v^2} \sum_n \frac{n(n+1)}{(\beta r)^2} I_n(r). \quad (139)$$

We have seen that as K increases indefinitely, all the complementary current waves approach zero; consequently the current-charge equations for the antenna become substantially the equations of a uniform transmission line. *If $I(r)$ is known*, we can determine RC and LC so that (139) would become identical with (134). *Assuming* that C is independent of r and is equal to the capacitance for the principal wave[44]

$$C = \frac{\pi\epsilon}{\log \cot \dfrac{\psi}{2}} = \frac{1}{vK}, \quad (140)$$

and comparing (134) and (139) we obtain

$$R + i\omega L = i\beta K - i\beta K \frac{\displaystyle\sum_n \frac{n(n+1)}{(\beta r)^2} I_n(r)}{I(r)}. \quad (141)$$

Taking $I(r)$ and $I_n(r)$ from the equations contained in the main section of the paper we can replace (141) by the following approximate expression:

[44] We must bear in mind that nothing can be deduced from (139) concerning the nature of C. In this respect the total-current—total-charge equations are unlike the principal-current—principal-charge equations.

$$R + i\omega L = i\beta K + \frac{60\pi \sum_{m=0}^{m} (4m + 3)[J_{2m+3/2}(\beta l) - iN_{2m+3/2}(\beta l)]\sqrt{rl}\, J_{2m+3/2}(\beta r)}{r^2\left[\sin \beta(l - r) + \dfrac{F - iG}{K} \cos \beta(l - r) + \dfrac{\bar{I}(r)}{I_0}\right]} + \text{remainder.} \qquad (142)$$

In order to obtain an expression for R when $l = \lambda/4$, we need only one term in the numerator and we may retain only the principal term in the denominator. Thus we shall have

$$R = \frac{360\left[\dfrac{\sin \beta r}{\beta r} - \cos \beta r\right]}{l(\beta r)^2 \cos \beta r}. \qquad (143)$$
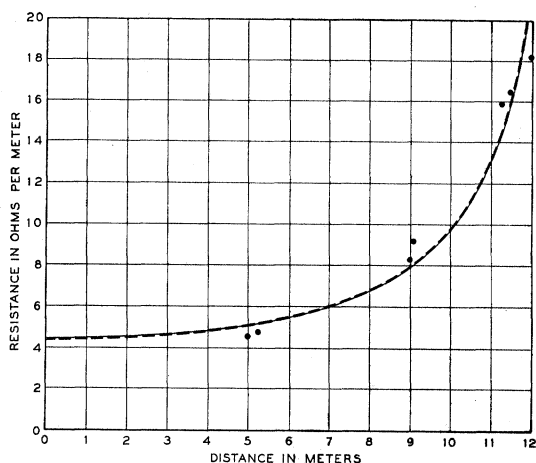

Fig. 29—Distribution of radiation resistance in the current-charge transmission equations. The calculated curve is for $2l = \lambda/2$; the experimental points, obtained from Feldman, are for $2l \doteq 2.5\lambda$.

In the neighborhood of the generator this becomes approximately

$$R = \frac{120}{l} = \frac{480}{\lambda}; \qquad (144)$$

and, near the ends,

$$R = \frac{5760}{\pi^4(l - r)} = \frac{59}{l - r}. \qquad (145)$$

The approximate expression for R obtained from $-E_z(z)/I(z)$, on the assumption that $I(z)$ is sinusoidal, is

$$R = \frac{120l}{(l^2 - r^2)}. \qquad (146)$$

This expression agrees quite well with ours (Fig. 29). However, there are no theoretical objections to (143) as there are to (146).

If the antenna is so short that βl is substantially smaller than unity and if we neglect those components of the total current which depend on K, then approximately

$$R = \frac{160\pi^2 l^2}{(l - r)\lambda^2}. \qquad (147)$$

The total radiated power calculated from (147) is found to agree with the well-known expression for the power radiated by a short doublet.

Besides introducing a distributed resistance into the current-charge equations of the antenna, radiation modifies the distributed inductance. Thus for $\beta l = \pi/2$ we find approximately (retaining only the first term in the summation)

$$\omega L = \beta K + 360\beta \frac{\dfrac{\sin \beta r}{\beta r} - \cos \beta r}{(\beta r)^2 \cos \beta r}. \qquad (148)$$

The first term of this expression represents the series reactance per unit length of an infinitely long conical antenna while the second term is the end effect. Near the center of the antenna (148) becomes

$$\omega L = \beta K + 120\beta. \qquad (149)$$

In the neighborhood of the ends of the antenna, the added reactance depends appreciably on other terms in (141); there the series converges slowly and even the approximation for the reactive part used by us in (142) is not good.

It is worth observing that if the series inductance is obtained from the imaginary part of $-E_z(z)/I(z)$ assuming $I(z)$ to be sinusoidal, the result

$$\omega L = \frac{120r}{l^2 - r^2} \tan \beta r \qquad (150)$$

is very disappointing and may be regarded as an additional reminder that the comparative success with the resistance must have been purely coincidental.

Thus from a theory in which antennas are regarded as multiple transmission lines, we have obtained two restricted theories. In the first of these radiation appears as a terminal impedance and in the second it appears as a distributed series impedance. There is no inconsistency between these two views and both are valid provided they are suitably qualified. Thus in the second theory, we think in terms of electric charge and electric current and must not assume the usual transmission-line relations between the voltage and the charge. In the first theory, on the other hand, all the usual transmission-line relations hold but the voltage, the charge, and the current in the transmission line representing the antenna correspond respectively to the voltage, the "principal" charge, and the principal current in the antenna.

VIII. Wave Propagation along Perfectly Conducting Parallel Wires

Consider a pair of parallel perfectly conducting wires (Fig. 30), energized by any number of "point generators." The electromagnetic field on or outside the wires can be expressed in the following general form

$$E = -i\omega\mu A - \text{grad } V, \quad H = \text{curl } A, \quad V = -\frac{\text{div } A}{i\omega\epsilon}, \quad (151)$$

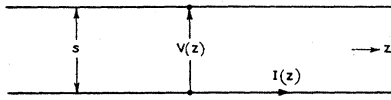where A is the retarded magnetic vector potential and V the retarded electric scalar potential.

Fig. 30—A diagram showing two perfectly conducting parallel wires.

If the wires are so energized that the electric conduction current is strictly longitudinal, and if the wires are thin, the current is always at least approximately longitudinal; the only nonvanishing component of A is the one parallel to the wires

$$A_x = A_y = 0, \qquad A_z = \Pi. \qquad (152)$$

In this case we obtain from (151)

$$E_z = -i\omega\mu\Pi - \frac{\partial V}{\partial z}, \quad V = -\frac{1}{i\omega\epsilon}\frac{\partial \Pi}{\partial z}. \quad (153)$$

Since the wires are perfectly conducting, E_z must vanish on the surface of each wire except at points of application of electromotive forces. Thus, we have

$$\frac{dV_1}{dz} = -i\omega\mu\Pi_1, \qquad \frac{d\Pi_1}{dz} = -i\omega\epsilon V_1,$$

$$\frac{dV_2}{dz} = -i\omega\mu\Pi_2, \qquad \frac{d\Pi_2}{dz} = -i\omega\epsilon V_2, \qquad (154)$$

where the subscripts designate the values of the corresponding functions on the wires. Subtracting, we have

$$\frac{d(V_1 - V_2)}{dz} = -i\omega\mu(\Pi_1 - \Pi_2),$$

$$\frac{d(\Pi_1 - \Pi_2)}{dz} = -i\omega\epsilon(V_1 - V_2). \qquad (155)$$

Thus, *the voltage across parallel perfectly conducting wires and the difference of vector potentials satisfy the equations of a uniform nondissipative transmission line. The phase velocity is the velocity of light.*

On the other hand, the electric current $I(z)$ does not satisfy the transmission-line equations. The vector potential Π is determined by the complete current distribution

$$\Pi(z) = \frac{1}{4\pi}\int \frac{I(z_1)e^{-i\beta r}}{r}dz_1, \qquad (156)$$

where r is the distance between the points z and z_1 and the integration is extended over both wires. If the distance between the wires is small compared with the

wavelength and with the length of the wires and if the wires are energized in "push-pull" so that the currents in the wires are equal and oppositely directed, then $\Pi_1 - \Pi_2$ is substantially proportional to the electric current at the corresponding point and (155) turns into the usual engineering equations governing transmission of waves on parallel wires.

The current in two thin infinitely long perfectly conducting parallel wires, energized in push-pull can be expressed in the following form[45]

$$I(z) = -\frac{i\beta a V_0}{2\eta\tau}\int_{(C)}\frac{\sinh \gamma\tau K_1(-ia\sqrt{\gamma^2 + \beta^2})e^{\gamma z}d\gamma}{\gamma\sqrt{\gamma^2 + \beta^2}[K_0(-is\sqrt{\gamma^2 + \beta^2} - K_0(-ia\sqrt{\gamma^2 + \beta^2})]}, \qquad (157)$$

where V_0 = an electromotive force distributed in push-pull in two short segments of length 2τ, one on each wire,

a = the radius of the wires,

s = the interaxial separation between the wires,

z = the distance from the mid-points of the "generators," and

C = the imaginary axis in the γ-plane indented around $\gamma = \pm i\beta$ (Fig. 31(A)).
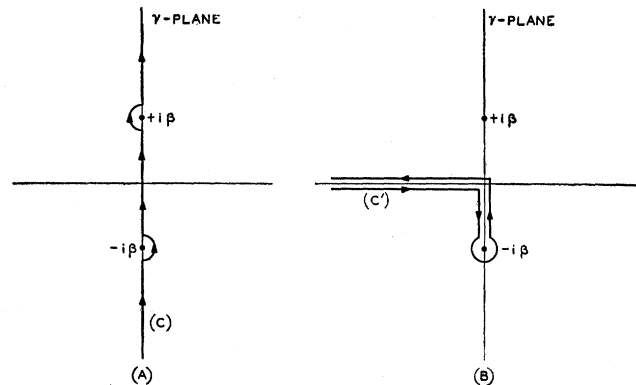

Fig. 31—Contours of integration.

The phase of $\sqrt{\gamma^2 + \beta^2}$ at $\gamma = \pm i\infty$ is $\pi/2$. The only approximation involved in (157) is that the radius of the wires be small compared with their interaxial separation; consequently, the "proximity effect" is not included in (157).

If z is positive, the contour (C) can be deformed into (C') as shown in Fig. 31(B). The integration around the point $\gamma = -i\beta$ which happens to be a pole as well as a branch point of the integrand, yields the following term

$$I_0(z) = \frac{V_0}{2K}\cdot\frac{\sin \beta\tau}{\beta\tau}e^{-i\beta z}, \quad K = \frac{\eta}{\pi}\log\frac{s}{a}, \quad z > 0. \quad (158)$$

This "principal" current wave is unattenuated and is substantially independent of τ if $2\pi\tau/\lambda$ is small

[45] In order to obtain this equation as well as equation (163) we express the "pulse function," representing the applied electromotive intensity $-E_z$, as a contour integral; then, we obtain the magnetic intensity H_ϕ as the quotient of $-E_z$ and the radial impedance; and finally we compute the current from $I(z) = 2\pi a H_\phi(a, z)$.

compared with unity. The remainder

$$I(z) = I(z) - I_0(z) \qquad (159)$$

is a purely local wave in the neighborhood of the generator and it represents the effect of radiation as well as the local reactance depending on τ.
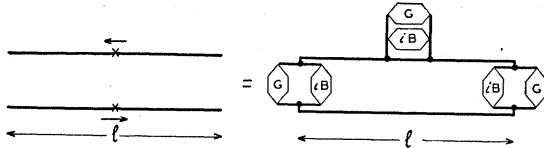

Fig. 32—The equivalent circuit for two parallel wires, open at both ends and energized in push-pull.

Radiation introduces a resistance in parallel with the applied electromotive force. If the interaxial separation is small compared with the wavelength, this resistance is approximately

$$R = \frac{K^2 \lambda^2}{60 \pi^2 s^2} . \qquad (160)$$

If the wires are of finite length, the expression for the current becomes vastly more complex. The result, however, will consist of two parts, one of which will represent the principal current wave and the other local waves in the neighborhood of the discontinuities, that is, ends of the wires or generators. It is fortunate, however, that the impedances introduced by radiation *at the points of discontinuity* can be calculated with good engineering accuracy without setting up exact equations but using in the proper manner the field produced by the principal current wave. The accuracy of such expressions increases with increasing characteristic impedance K.

Let a pair of parallel wires of radius a, whose interaxial separation is s, be open at both ends and be energized in "push-pull"; then, in free space, the radiation can be represented by parallel combinations of conductance G and susceptance $B = \omega C$ as shown in Fig. 32, where

$$G = \frac{60\pi^2(s^2 - a^2)}{\lambda^2}, \qquad C = \frac{60(s - a)}{vK^2},$$

$$B = \omega C = \frac{120\pi(s - a)}{K^2 \lambda} . \qquad (161)$$

We omit the details of calculation.

If the wires are imperfect conductors, the first equation of the set (155) becomes

$$\frac{d(V_1 - V_2)}{dz} = - i\omega\mu(\Pi_1 - \Pi_2) - Z_1 I_1 - Z_2 I_2, \qquad (162)$$

where Z_1 and Z_2 are the surface impedances of the wires. The added terms affect the voltage across the pair continuously. The principal effect is due to dissipation of energy in the wires; but since the current

distribution is affected by radiation, the latter also will produce a continuously distributed effect on the voltage. The latter is, however, a second-order effect; besides, it is concentrated largely in the immediate vicinity of the ends, even though mathematically speaking it extends over the entire length.

IX. ON THE IMPEDANCE OF AN INFINITELY LONG PERFECTLY CONDUCTING CYLINDRICAL WIRE

It is readily shown[46] that the current in an infinitely long perfectly conducting cylindrical wire of radius a is

$$I(z) = \frac{i\beta a V_0}{\eta \tau} \int_{(C)} \frac{\sinh \gamma\tau \, K_1(-ia\sqrt{\gamma^2+\beta^2})}{\gamma\sqrt{\gamma^2+\beta^2} K_0(-ia\sqrt{\gamma^2+\beta^2})} e^{\gamma z} d\gamma, \qquad (163)$$

where V_0 is the applied electromotive force uniformly distributed over a section of length τ and C is the contour shown in Fig. 31(A).

Deforming (C) into (C'), we can show that the real component of the input admittance is substantially independent of τ and for $\tau = 0$ is

$$\frac{1}{K(0)} = \frac{4\beta}{\eta\pi} \int_0^\beta \frac{d\xi}{(\beta^2 - \xi^2)[J_0{}^2(a\sqrt{\beta^2 - \xi^2}) + N_0{}^2(a\sqrt{\beta^2 - \xi^2})]} . \qquad (164)$$

After some tedious transformations we have obtained the following approximate expression (when $\eta = 120\pi$) for thin wires

$$K(0) = \frac{120M}{1 + \dfrac{\log 2}{2M} - \dfrac{\frac{2}{3}\pi^2 - (\log 2)^2}{4M^2}}$$

$$M = \log \frac{\lambda}{2\pi a} + \tfrac{1}{2} \log 2 - C, \qquad (165)$$

where $C = 0.577 \cdots$ is Euler's constant. As $a \to 0$, (165) approaches

$$K(0) = 120\left(\log\frac{\lambda}{2\pi a} - C\right) = 120 \log\frac{\lambda}{2a} - 207. \qquad (166)$$

X. AN APPROXIMATION OF A DISSIPATIVE TRANSMISSION LINE BY A NONDISSIPATIVE LINE WITH AN EFFECTIVE LOAD AT THE LAST CURRENT NODE OR ANTINODE

The input impedance of a dissipative line of length l, electrically open at the far end, is

$$Z_i = K \coth (\alpha + i\beta)l, \qquad (167)$$

where $\alpha + i\beta$ is the propagation constant. If αl is small compared with unity, then we have approximately

$$Z_i = K \frac{\cos \beta l + i\alpha l \sin \beta l}{\alpha l \cos \beta l + i \sin \beta l} . \qquad (168)$$

For a nondissipative line with a terminal impedance Z_t at the far end, we have

[46] See footnote 45.

$$Z_i = K \frac{Z_t \cos \beta l + iK \sin \beta l}{K \cos \beta l + iZ_t \sin \beta l}. \qquad (169)$$

Multiplying the numerator and the denominator of (168) by $K/\alpha l$ and comparing with (169) we obtain

$$Z_t = \frac{K}{\alpha l} = \frac{K^2}{\frac{1}{2}Rl}, \qquad (170)$$

where R is the resistance per unit length of the dissipative line.

The same input impedance will be obtained, of course, if the nondissipative line is electrically open and at the point $\frac{1}{4}$ wavelength from the open end there is an impedance Z, in series with the line, defined by

$$Z = \frac{K^2}{Z_t} = \frac{\alpha l}{K} = \frac{1}{2}Rl. \qquad (171)$$

It is to be expected, therefore, that the electric currents in these three different cases will be approximately the same in the neighborhood of the generator. If the line is about $\frac{1}{2}$ wavelength long, the minimum current will occur close to the generator and its measured value will not enable us to decide which, if any, of the three above-discussed distributions of loss happens to be the true distribution. At the current antinode, the electric currents in the three cases differ by larger absolute amounts; these differences are, however, small compared with the total current.

High-Frequency Radio Transmission Conditions, August, 1941, with Predictions for November, 1941*

NATIONAL BUREAU OF STANDARDS, WASHINGTON, D. C.

Note: In order to make these monthly reports of maximum service, changes are made from time to time. The Bureau welcomes suggestions for improvement of both form and substance. Address any communications to the National Bureau of Standards, Washington, D. C.

THE radio transmission data herein are based on observations at Washington, D. C., of long-distance reception and of the ionosphere. Fig. 1 gives the August average values of maximum usable frequencies, for undisturbed days, for radio transmission by way of the regular layers of the ionosphere. The regular-layer maximum usable frequencies were determined by the F layer at night and by the E, F_1,
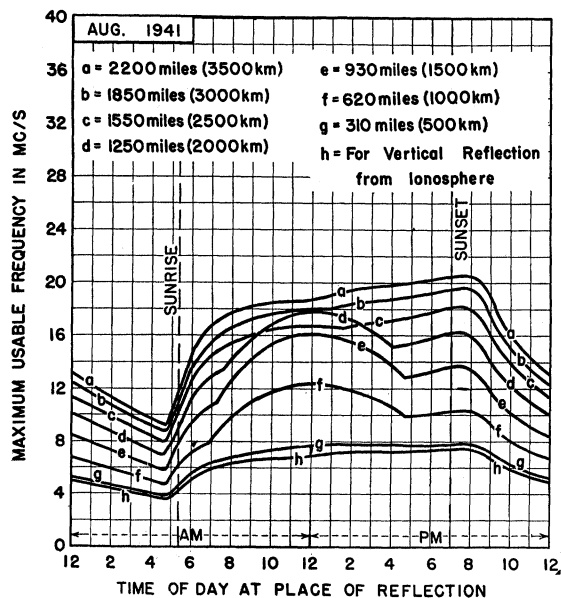

Fig. 1—Maximum usable frequencies for dependable radio transmission via the regular layers, average for undisturbed days, for August, 1941. The values shown were considerably exceeded during frequent irregular periods by reflections from clouds of sporadic E layer. (See Table II.) These curves and those of Fig. 2 also give skip distances, since the maximum usable frequency for a given distance is the frequency for which that distance is the skip distance.
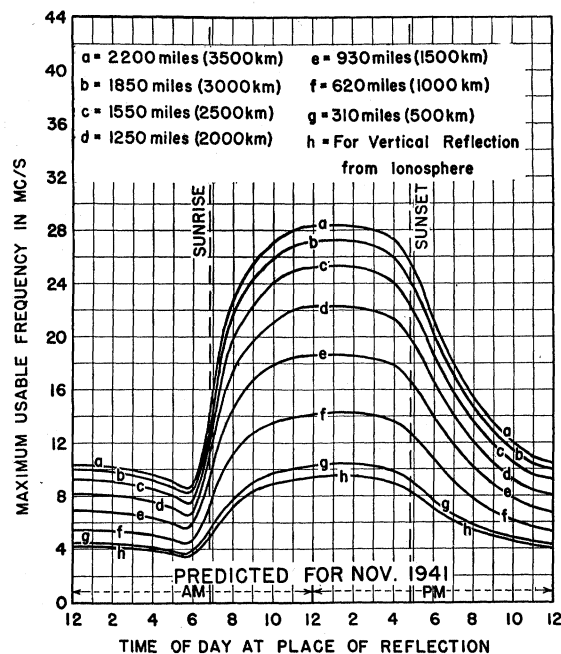

Fig. 2—Predicted maximum usable frequencies for dependable radio transmission via the regular layers average for undisturbed days, for November, 1941. For information on use in practical radio transmission problems, see the pamphlets "Radio transmission and the ionosphere" and "Distance ranges of radio waves," obtainable from The National Bureau of Standards, Washington, D. C., on request.

and F_2 layers during the day. Fig. 2 gives the expected values of the maximum usable frequencies for radio transmission by way of the regular layers, average for

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