Appendix:
TABLE OF INTEGRALS

148. In the following table, the constant of integration, C, is omitted but should be added to the result of every integration. The letter x represents any variable; u represents any function of x; the remaining letters represent arbitrary constants, unless otherwise indicated; all angles are in radians. Unless otherwise mentioned, $\log_e u = \log u$.

Expressions Containing $ax + b$

1. $\int (ax + b)^n \, dx = \frac{1}{a(n + 1)}(ax + b)^{n+1}, \quad n \neq -1.$

2. $\int \frac{dx}{ax + b} = \frac{1}{a} \log_e(ax + b).$

3. $\int \frac{dx}{(ax + b)^2} = -\frac{1}{a(ax + b)}.$

4. $\int \frac{dx}{(ax + b)^3} = -\frac{1}{2a(ax + b)^2}.$

5. $\int x(ax + b)^n \, dx = \frac{1}{a^2(n + 2)}(ax + b)^{n+2} - \frac{b}{a^2(n + 1)}(ax + b)^{n+1}, \quad n \neq -1, -2.$

6. $\int \frac{x \, dx}{ax + b} = \frac{x}{a} - \frac{b}{a^2} \log(ax + b).$

7. $\int \frac{x \, dx}{(ax + b)^2} = \frac{b}{a^2(ax + b)} + \frac{1}{a^2} \log(ax + b).$

8. $$\int \frac{x\,dx}{(ax+b)^3} = \frac{b}{2a^2(ax+b)^2} - \frac{1}{a^2(ax+b)}.$$

9. $$\int x^2(ax+b)^n\,dx = \frac{1}{a^2}\left(\frac{(ax+b)^{n+3}}{n+3} - 2b\frac{(ax+b)^{n+2}}{n+2} + \frac{b^2(ax+b)^{n+1}}{n+1}\right), \quad n \neq -1, -2, -3.$$

10. $$\int \frac{x^2\,dx}{ax+b} = \frac{1}{a^3}\left(\frac{1}{2}(ax+b)^2 - 2b(ax+b) + b^2\log(ax+b)\right).$$

11. $$\int \frac{x^2\,dx}{(ax+b)^2} = \frac{1}{a^3}\left((ax+b) - 2b\log(ax+b) - \frac{b^2}{ax+b}\right).$$

12. $$\int \frac{x^2\,dx}{(ax+b)^3} = \frac{1}{a^3}\left(\log(ax+b) + \frac{2b}{ax+b} - \frac{b^2}{2(ax+b)^2}\right).$$

13. $$\int x^m(ax+b)^n\,dx = \frac{1}{a(m+n+1)}\left(x^m(ax+b)^{n+1} - mb\int x^{m-1}(ax+b)^n\,dx\right)$$
$$= \frac{1}{m+n+1}\left(x^{m+1}(ax+b)^n + nb\int x^m(ax+b)^{n-1}\,dx\right),$$
$$m > 0,\ m+n+1 \neq 0.$$

14. $$\int \frac{dx}{x(ax+b)} = \frac{1}{b}\log\left(\frac{x}{ax+b}\right).$$

15. $$\int \frac{dx}{x^2(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2}\log\left(\frac{ax+b}{x}\right).$$

16. $$\int \frac{dx}{x^3(ax+b)} = \frac{2ax-b}{2b^2x^2} + \frac{a^2}{b^3}\log\left(\frac{x}{ax+b}\right).$$

17. $$\int \frac{dx}{x(ax+b)^2} = \frac{1}{b(ax+b)} - \frac{1}{b^2}\log\left(\frac{ax+b}{x}\right).$$

18. $$\int \frac{dx}{x(ax+b)^3} = \frac{1}{b^3}\left[\frac{1}{2}\left(\frac{ax+2b}{ax+b}\right)^2 + \log\left(\frac{x}{ax+b}\right)\right].$$

19. $$\int \frac{dx}{x^2(ax+b)^2} = -\frac{b+2\,ax}{b^2x(ax+b)} + \frac{2a}{b^3}\log\left(\frac{ax+b}{x}\right).$$

20. $$\int \sqrt{ax+b}\,dx = \frac{2}{3a}\sqrt{(ax+b)^3}.$$

21. $$\int x\sqrt{ax+b}\,dx = \frac{2(3ax-2b)}{15a^2}\sqrt{(ax+b)^3}.$$

22. $$\int x^2\sqrt{ax+b}\,dx = \frac{2(15a^2x^2 - 12abx + 8b^2)\sqrt{(ax+b)^3}}{105a^3}$$

23. $$\int x^3\sqrt{ax+b}\,dx = \frac{2(35a^3x^3 - 30a^2bx^2 + 24ab^2x - 16b^3)\sqrt{(ax+b)^3}}{315a^4}.$$

24. $\displaystyle\int x^n\sqrt{ax+b}\,dx = \frac{2}{a^{n+1}}\int u^2(u^2-b)^n\,du, \qquad u=\sqrt{ax+b}.$

25. $\displaystyle\int\frac{\sqrt{ax+b}}{x}\,dx = 2\sqrt{ax+b}+b\int\frac{dx}{x\sqrt{ax+b}}.$

26. $\displaystyle\int\frac{dx}{\sqrt{ax+b}} = \frac{2\sqrt{ax+b}}{a}.$

27. $\displaystyle\int\frac{x\,dx}{\sqrt{ax+b}} = \frac{2(ax-2b)}{3a^2}\sqrt{ax+b}.$

28. $\displaystyle\int\frac{x^2\,dx}{\sqrt{ax+b}} = \frac{2(3a^2x^2-4abx+8b^2)}{15a^3}\sqrt{ax+b}.$

29. $\displaystyle\int\frac{x^3\,dx}{\sqrt{ax+b}} = \frac{2(5a^3x^3-6a^2bx^2+8ab^2x-16b^3)}{35a^4}\sqrt{ax+b}.$

30. $\displaystyle\int\frac{x^n\,dx}{\sqrt{ax+b}} = \frac{2}{a^{n+1}}\int(u^2-b)^n\,du, \qquad u=\sqrt{ax+b}.$

31. $\displaystyle\int\frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\log\left(\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right) \quad \text{for } b>0.$

32. $\displaystyle\int\frac{dx}{x\sqrt{ax+b}} = \frac{2}{\sqrt{-b}}\tan^{-1}\sqrt{\frac{ax+b}{-b}}, \quad b<0$

$$= \frac{-2}{\sqrt{b}}\tanh^{-1}\sqrt{\frac{ax+b}{-b}}, \quad b>0.$$

33. $\displaystyle\int\frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx}-\frac{a}{2b}\int\frac{dx}{x\sqrt{ax+b}}.$

34. $\displaystyle\int\frac{dx}{x^3\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{2bx^2}+\frac{3a\sqrt{ax+b}}{4b^2x}+\frac{3a^2}{8b^2}\int\frac{dx}{x\sqrt{ax+b}}.$

35. $\displaystyle\int\frac{dx}{x^n(ax+b)^m} = -\frac{1}{b^{m+n-1}}\int\frac{(u-a)^{m+n-2}\,du}{u^m}, \qquad u=\frac{ax+b}{x}.$

36. $\displaystyle\int(ax+b)^{\pm n/2}\,dx = \frac{2(ax+b)^{(2\pm n)/2}}{a(2\pm n)}.$

37. $\displaystyle\int x(ax+b)^{\pm n/2}\,dx = \frac{2}{a^2}\left(\frac{(ax+b)^{(4\pm n)/2}}{4\pm n}-\frac{b(ax+b)^{(2\pm n)/2}}{2\pm n}\right).$

38. $\displaystyle\int\frac{dx}{x(ax+b)^{n/2}} = \frac{1}{b}\int\frac{dx}{x(ax+b)^{(n-2)/2}}-\frac{a}{b}\int\frac{dx}{(ax+b)^{n/2}}.$

39. $\displaystyle\int\frac{x^m\,dx}{\sqrt{ax+b}} = \frac{2x^m\sqrt{ax+b}}{(2m+1)a}-\frac{2mb}{(2m+1)a}\int\frac{x^{m-1}\,dx}{\sqrt{ax+b}}.$

40. $\displaystyle\int\frac{dx}{x^n\sqrt{ax+b}} = \frac{-\sqrt{ax+b}}{(n-1)bx^{n-1}}-\frac{(2n-3)a}{(2n-2)b}\int\frac{dx}{x^{n-1}\sqrt{ax+b}}.$

41. $\displaystyle\int\frac{(ax+b)^{n/2}}{x}\,dx = a\int(ax+b)^{(n-2)/2}\,dx+b\int\frac{(ax+b)^{(n-2)/2}}{x}\,dx.$

42. $$\int \frac{dx}{(ax+b)(cx+d)} = \frac{1}{bc-ad}\log\left(\frac{cx+d}{ax+b}\right), \qquad bc-ad \neq 0.$$

43. $$\int \frac{dx}{(ax+b)^2(cx+d)} = \frac{1}{bc-ad}\left[\frac{1}{ax+b} + \frac{c}{bc-ad}\log\left(\frac{cx+d}{ax+b}\right)\right], \qquad bc-ad \neq 0.$$

44. $$\int (ax+b)^n(cx+d)^m\,dx = \frac{1}{(m+n+1)a}\left[(ax+b)^{n+1}(cx+d)^m - m(bc-ad)\int (ax+b)^n(cx+d)^{m-1}\,dx\right].$$

45. $$\int \frac{dx}{(ax+b)^n(cx+d)^m} = \frac{1}{(m-1)(bc-ad)}\left(\frac{1}{(ax+b)^{n-1}(cx+d)^{m-1}} - a(m+n-2)\int \frac{dx}{(ax+b)^n(cx+d)^{m-1}}\right), \qquad m>1,\ n>0,\ bc-ad\neq 0.$$

46. $$\int \frac{(ax+b)^n}{(cx+d)^m}\,dx$$
$$= -\frac{1}{(m-1)(bc-ad)}\left(\frac{(ax+b)^{n+1}}{(cx+d)^{m-1}} + (m-n-2)a\int \frac{(ax+b)^n dx}{(cx+d)^{m-1}}\right)$$
$$= \frac{1}{(m-n-1)c}\left(\frac{(ax+b)^n}{(cx+d)^{m-1}} + n(bc-ad)\int \frac{(ax+b)^{n-1}}{(cx+d)^m}\,dx\right).$$

47. $$\int \frac{x\,dx}{(ax+b)(cx+d)} = \frac{1}{bc-ad}\left(\frac{b}{a}\ln|ax+b| - \frac{d}{c}\log(cx+d)\right), \qquad bc-ad\neq 0.$$

48. $$\int \frac{x\,dx}{(ax+b)^2(cx+d)} = \frac{1}{bc-ad}\left(-\frac{b}{a(ax+b)} - \frac{d}{bc-ad}\log\left(\frac{cx+d}{ax+b}\right)\right), \qquad bc-ad\neq 0.$$

49. $$\int \frac{cx+d}{\sqrt{ax+b}}\,dx = \frac{2}{3a^2}(3ad-2bc+acx)\sqrt{ax+b}.$$

50. $$\int \frac{\sqrt{ax+b}}{cx+d}\,dx = \frac{2\sqrt{ax+b}}{c} - \frac{2}{c}\sqrt{\frac{ad-bc}{c}}\tan^{-1}\sqrt{\frac{c(ax+b)}{ad-bc}}, \qquad c>0,\ ad>bc.$$

51. $$\int \frac{\sqrt{ax+b}}{cx+d}\,dx = \frac{2\sqrt{ax+b}}{c} + \frac{1}{c}\sqrt{\frac{bc-ad}{c}}\log\left(\frac{\sqrt{c(ax+b)}-\sqrt{bc-ad}}{\sqrt{c(ax+b)}+\sqrt{bc-ad}}\right), \qquad c>0,\ bc>ad.$$

52. $$\int \frac{dx}{(cx+d)\sqrt{ax+b}} = \frac{2}{\sqrt{c}\sqrt{ad-bc}}\tan^{-1}\sqrt{\frac{c(ax+b)}{ad-bc}}, \qquad c>0,\ ad>bc.$$

53. $$\int \frac{dx}{(cx+d)\sqrt{ax+b}} = \frac{1}{\sqrt{c}\sqrt{bc-ad}} \log\left(\frac{\sqrt{c(ax+b)}-\sqrt{bc-ad}}{\sqrt{c(ax+b)}+\sqrt{bc-ad}}\right), \qquad c>0,\ bc>ad.$$

Expressions Containing $ax^2 + c$, $ax^n + c$, $x^2 \pm p^2$, and $p^2 - x^2$

54. $$\int \frac{dx}{p^2+x^2} = \frac{1}{p}\tan^{-1}\frac{x}{p} \quad \text{or} \quad -\frac{1}{p}\operatorname{ctn}^{-1}\left(\frac{x}{p}\right).$$

55. $$\int \frac{dx}{p^2-x^2} = \frac{1}{2p}\log\left(\frac{p+x}{p-x}\right), \quad \text{or} \quad \frac{1}{p}\tanh^{-1}\left(\frac{x}{p}\right).$$

56. $$\int \frac{dx}{ax^2+c} = \frac{1}{\sqrt{ac}}\tan^{-1}\left(x\sqrt{\frac{a}{c}}\right), \qquad a \text{ and } c>0.$$

57. $$\int \frac{dx}{ax^2+c} = \frac{1}{2\sqrt{-ac}}\log\left(\frac{x\sqrt{a}-\sqrt{-c}}{x\sqrt{a}+\sqrt{-c}}\right), \qquad n>0,\ c<0$$
$$= \frac{1}{2\sqrt{-ac}}\log\left(\frac{\sqrt{c}+x\sqrt{-a}}{\sqrt{c}-x\sqrt{-a}}\right), \qquad a<0,\ c>0.$$

58. $$\int \frac{dx}{(ax^2+c)^n} = \frac{1}{2(n-1)c}\frac{x}{(ax^2+c)^{n-1}} + \frac{2n-3}{2(n-1)c}\int \frac{dx}{(ax^2+c)^{n-1}}, \qquad n \text{ a positive integer.}$$

59. $$\int x(ax^2+c)^n\,dx = \frac{1}{2a}\frac{(ax^2+c)^{n+1}}{n+1}, \qquad n \neq -1.$$

60. $$\int \frac{x}{ax^2+c}\,dx = \frac{1}{2a}\log(ax^2+c).$$

61. $$\int \frac{dx}{x(ax^2+c)} = \frac{1}{2c}\log\left(\frac{ax^2}{ax^2+c}\right).$$

62. $$\int \frac{dx}{x^2(ax^2+c)} = -\frac{1}{cx} - \frac{a}{c}\int \frac{dx}{ax^2+c}.$$

63. $$\int \frac{x^2\,dx}{ax^2+c} = \frac{x}{a} - \frac{c}{a}\int \frac{dx}{ax^2+c}.$$

64. $$\int \frac{x^n\,dx}{ax^2+c} = \frac{x^{n-1}}{a(n-1)} - \frac{c}{a}\int \frac{x^{n-2}\,dx}{ax^2+c}, \qquad n \neq 1.$$

65. $$\int \frac{x^2\,dx}{(ax^2+c)^n} = -\frac{1}{2(n-1)a}\frac{x}{(ax^2+c)^{n-1}} + \frac{1}{2(n-1)a}\int \frac{dx}{(ax^2+c)^{n-1}}.$$

66. $\displaystyle\int \frac{dx}{x^2(ax^2+c)^n} = \frac{1}{c}\int \frac{dx}{x^2(ax^2+c)^{n-1}} - \frac{a}{c}\int \frac{dx}{(ax^2+c)^n}.$

67. $\displaystyle\int \sqrt{x^2 \pm p^2}\, dx = \frac{1}{2}[x\sqrt{x^2 \pm p^2} \pm p^2 \log(x + \sqrt{x^2 \pm p^2})].$

68. $\displaystyle\int \sqrt{p^2 - x^2}\, dx = \frac{1}{2}\left[x\sqrt{p^2 - x^2} + p^2 \sin^{-1}\left(\frac{x}{p}\right)\right].$

69. $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm p^2}} = \log(x + \sqrt{x^2 \pm p^2}).$

70. $\displaystyle\int \frac{dx}{\sqrt{p^2 - x^2}} = \sin^{-1}\left(\frac{x}{p}\right) \quad \text{or} \quad -\cos^{-1}\left(\frac{x}{p}\right).$

71. $\displaystyle\int \sqrt{ax^2 + c}\, dx = \frac{x}{2}\sqrt{ax^2 + c} + \frac{c}{2\sqrt{a}} \log(x\sqrt{a} + \sqrt{ax^2 + c}), \qquad a > 0.$

72. $\displaystyle\int \sqrt{ax^2 + c}\, dx = \frac{x}{2}\sqrt{ax^2 + c} + \frac{c}{2\sqrt{-a}} \sin^{-1}\left(x\sqrt{\frac{-a}{c}}\right), \qquad a < 0.$

73. $\displaystyle\int \frac{dx}{\sqrt{ax^2 + c}} = \frac{1}{\sqrt{a}} \log(x\sqrt{a} + \sqrt{ax^2 + c}), \qquad a > 0.$

74. $\displaystyle\int \frac{dx}{\sqrt{ax^2 + c}} = \frac{1}{\sqrt{-a}} \sin^{-1}\left(x\sqrt{\frac{-a}{c}}\right), \qquad a < 0.$

75. $\displaystyle\int x\sqrt{ax^2 + c}\, dx = \frac{1}{3a}(ax^2 + c)^{3/2}.$

76. $\displaystyle\int x^2\sqrt{ax^2 + c}\, dx = \frac{x}{4a}\sqrt{(ax^2 + c)^3} - \frac{cx}{8a}\sqrt{ax^2 + c} - \frac{c^2}{8\sqrt{a^3}} \log(x\sqrt{a} + \sqrt{ax^2 + c}), \qquad a > 0.$

77. $\displaystyle\int x^2\sqrt{ax^2 + c}\, dx = \frac{x}{4a}\sqrt{(ax^2 + c)^3} - \frac{cx}{8a}\sqrt{ax^2 + c} - \frac{c^2}{8a\sqrt{-a}} \sin^{-1}\left(x\sqrt{\frac{-a}{c}}\right), \qquad a < 0.$

78. $\displaystyle\int \frac{x\, dx}{\sqrt{ax^2 + c}} = \frac{1}{a}\sqrt{ax^2 + c}.$

79. $\displaystyle\int \frac{x^2\, dx}{\sqrt{ax^2 + c}} = \frac{x}{a}\sqrt{ax^2 + c} - \frac{1}{a}\int \sqrt{ax^2 + c}\, dx.$

80. $\displaystyle\int \frac{\sqrt{ax^2 + c}}{x}\, dx = \sqrt{ax^2 + c} + \sqrt{c} \log\left(\frac{\sqrt{ax^2 + c} - \sqrt{c}}{x}\right), \qquad c > 0.$

81. $\displaystyle\int \frac{\sqrt{ax^2 + c}}{x}\, dx = \sqrt{ax^2 + c} - \sqrt{-c} \tan^{-1}\frac{\sqrt{ax^2 + c}}{\sqrt{-c}}, \qquad c < 0.$

82. $\displaystyle\int \frac{dx}{x\sqrt{p^2 \pm x^2}} = -\frac{1}{p} \log\left(\frac{p + \sqrt{p^2 \pm x^2}}{x}\right).$

83. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - p^2}} = \frac{1}{p}\cos^{-1}\left(\frac{p}{x}\right) \quad \text{or} \quad -\frac{1}{p}\sin^{-1}\left(\frac{p}{x}\right).$

84. $\displaystyle\int \frac{dx}{x\sqrt{ax^2 + c}} = \frac{1}{\sqrt{c}}\log\left(\frac{\sqrt{ax^2 + c} - \sqrt{c}}{x}\right), \qquad c > 0.$

85. $\displaystyle\int \frac{dx}{x\sqrt{ax^2 + c}} = \frac{1}{\sqrt{-c}}\sec^{-1}\left(x\sqrt{-\frac{a}{c}}\right), \qquad c < 0.$

86. $\displaystyle\int \frac{dx}{x^2\sqrt{ax^2 + c}} = -\frac{\sqrt{ax^2 + c}}{cx}.$

87. $\displaystyle\int \frac{x^n\,dx}{\sqrt{ax^2 + c}} = \frac{x^{n-1}\sqrt{ax^2 + c}}{na} - \frac{(n-1)c}{na}\int \frac{x^{n-2}\,dx}{\sqrt{ax^2 + c}}, \qquad n > 0.$

88. $\displaystyle\int x^n\sqrt{ax^2 + c}\,dx = \frac{x^{n-1}(ax^2 + c)^{3/2}}{(n+2)a} - \frac{(n-1)c}{(n+2)a}\int x^{n-2}\sqrt{ax^2 + c}\,dx, \qquad n > 0.$

89. $\displaystyle\int \frac{\sqrt{ax^2 + c}}{x^n}\,dx = -\frac{(ax^2 + c)^{3/2}}{c(n-1)x^{n-1}} - \frac{(n-4)a}{(n-1)c}\int \frac{\sqrt{ax^2 + c}}{x^{n-2}}\,dx, \qquad n > 1.$

90. $\displaystyle\int \frac{dx}{x^n\sqrt{ax^2 + c}} = -\frac{\sqrt{ax^2 + c}}{c(n-1)x^{n-1}} - \frac{(n-2)\,a}{(n-1)\,c}\int \frac{dx}{x^{n-2}\sqrt{ax^2 + c}}, \qquad n > 1.$

91. $\displaystyle\int (ax^2 + c)^{3/2}\,dx = \frac{x}{8}(2ax^2 + 5c)\sqrt{ax^2 + c} + \frac{3c^2}{8\sqrt{a}}\log(x\sqrt{a} + \sqrt{ax^2 + c}), \qquad a > 0.$

92. $\displaystyle\int (ax^2 + c)^{3/2}\,dx = \frac{x}{8}(2ax^2 + 5c)\sqrt{ax^2 + c} + \frac{3c^2}{8\sqrt{-a}}\sin^{-1}\left(x\sqrt{\frac{-a}{c}}\right), \qquad a < 0.$

93. $\displaystyle\int \frac{dx}{(ax^2 + c)^{3/2}} = \frac{x}{c\sqrt{ax^2 + c}}.$

94. $\displaystyle\int x(ax^2 + c)^{3/2}\,dx = \frac{1}{5a}(ax^2 + c)^{3/2}.$

95. $\displaystyle\int x^2(ax^2 + c)^{3/2}\,dx = \frac{x^3}{6}(ax^2 + c)^{3/2} + \frac{c}{2}\int x^2\sqrt{ax^2 + c}\,dx.$

96. $\displaystyle\int x^n(ax^2 + c)^{3/2}\,dx = \frac{x^{n+1}(ax^2 + c)^{3/2}}{n+4} + \frac{3c}{n+4}\int x^n\sqrt{ax^2 + c}\,dx.$

97. $$\int \frac{x\,dx}{(ax^2+c)^{3/2}} = -\frac{1}{a\sqrt{ax^2+c}}.$$

98. $$\int \frac{x^2\,dx}{(ax^2+c)^{3/2}} = -\frac{x}{a\sqrt{ax^2+c}} + \frac{1}{a\sqrt{a}}\log(x\sqrt{a}+\sqrt{ax^2+c}), \qquad a>0.$$

99. $$\int \frac{x^2\,dx}{(ax^2+c)^{3/2}} = -\frac{x}{a\sqrt{ax^2+c}} + \frac{1}{a\sqrt{-a}}\sin^{-1}\left(x\sqrt{\frac{-a}{c}}\right), \qquad a<0.$$

100. $$\int \frac{x^3\,dx}{(ax^2+c)^{3/2}} = -\frac{x^2}{a\sqrt{ax^2+c}} + \frac{2}{a^2}\sqrt{ax^2+c}.$$

101. $$\int \frac{dx}{x(ax^n+c)} = \frac{1}{cn}\log\left(\frac{x^n}{ax^n+c}\right).$$

102. $$\int \frac{dx}{(ax^n+c)^m} = \frac{1}{c}\int \frac{dx}{(ax^n+c)^{m-1}} - \frac{a}{c}\int \frac{x^n\,dx}{(ax^n+c)^m}.$$

103. $$\int \frac{dx}{x\sqrt{ax^n+c}} = \frac{1}{n\sqrt{c}}\log\left(\frac{\sqrt{ax^n+c}-\sqrt{c}}{\sqrt{ax^n+c}+\sqrt{c}}\right), \qquad c>0.$$

104. $$\int \frac{dx}{x\sqrt{ax^n+c}} = \frac{2}{n\sqrt{-c}}\sec^{-1}\sqrt{\frac{-ax^n}{c}}, \qquad c<0.$$

105. $$\int x^{m-1}(ax^n+c)^p\,dx$$
$$= \frac{1}{m+np}[x^m(ax^n+c)^p + npc\int x^{m-1}(ax^n+c)^{p-1}\,dx]$$
$$= \frac{1}{cn(p+1)}[-x^m(ax^n+c)^{p+1} + (nt+np+n)\int x^{m-1}(ax^n+c)^{p+1}\,dx].$$
$$= \frac{1}{a(m+np)}[x^{m-n}(ax^n+c)^{p'+1} - (m-n)c\int x^{m-n-1}(ax^n+c)^p\,dx].$$
$$= \frac{1}{mc}[x^m(ax^n+c)^{p+1} - (m+np+n)\int x^{m+n-1}(ax^n+c)^p\,dx].$$

106. $$\int \frac{x^m dx}{(ax^n+c)^p} = \frac{1}{a}\int \frac{x^{m-n}\,dx}{(ax^n+c)^{p-1}} - \frac{c}{a}\int \frac{x^{m-n}\,dx}{(ax^n+c)^p}.$$

107. $$\int \frac{dx}{x^m(ax^n+c)^p} = \frac{1}{c}\int \frac{dx}{x^m(ax^n+c)^{p-1}} - \frac{a}{c}\int \frac{dx}{x^{m-n}(ax^n+c)^p}.$$

Expressions Containing $ax^2 + bx + c$

108. $$\int \frac{dx}{ax^2+bx+c} = \frac{1}{\sqrt{b^2-4ac}}\log\left(\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right), \qquad b^2>4ac.$$

109. $$\int \frac{dx}{ax^2+bx+c} = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}}, \quad b^2 < 4ac.$$

110. $$\int \frac{dx}{ax^2+bx+c} = -\frac{2}{2\,ax+b}, \qquad b^2 = 4ac.$$

111. $$\int \frac{dx}{(ax^2+bx+c)^{n+1}} = \frac{2ax+b}{n(4ac-b^2)(ax^2+bx+c)^n} + \frac{2(2n-1)a}{n(4ac-b^2)} \int \frac{dx}{(ax^2+bx+c)^n}.$$

112. $$\int \frac{x\,dx}{ax^2+bx+c} = \frac{1}{2a} \log(ax^2+bx+c) - \frac{b}{2a} \int \frac{dx}{ax^2+bx+c}.$$

113. $$\int \frac{x^2\,dx}{ax^2+bx+c} = \frac{x}{a} - \frac{b}{2a^2} \log(ax^2+bx+c) + \frac{b^2-2\,ac}{2\,a^2} \int \frac{dx}{ax^2+bx+c}.$$

114. $$\int \frac{x^n dx}{ax^2+bx+c} = \frac{x^{n-1}}{(n-1)a} - \frac{c}{a} \int \frac{x^{n-2}\,dx}{ax^2+bx+c} - \frac{b}{a} \int \frac{x^{n-1}\,dx}{ax^2+bx+c}.$$

115. $$\int \frac{xdx}{(ax^2+bx+c)^{n+1}} = \frac{-(2c+bx)}{n(4\,ac-b^2)(ax^2+bx+c)^n} - \frac{b(2n-1)}{n(4\,ac-b^2)} \int \frac{dx}{(ax^2+bx+c)^n}.$$

116. $$\int \frac{x^m\,dx}{(ax^2+bx+c)^{n+1}} = -\frac{x^{m-1}}{a(2n-m+1)(ax^2+bx+c)^n} - \frac{n-m+1}{2n-m+1} \frac{b}{a} \int \frac{x^{m-1}\,dx}{(ax^2+bx+c)^{n+1}} + \frac{m-1}{2n-m+1} \frac{c}{a} \int \frac{x^{m-2}\,dx}{(ax^2+bx+c)^{n+1}}$$

117. $$\int \frac{dx}{x(ax^2+bx+c)} = \frac{1}{2c} \log\left(\frac{x^2}{ax^2+bx+c}\right) - \frac{b}{2c} \int \frac{dx}{(ax^2+bx+c)}.$$

118. $$\int \frac{dx}{x^2(ax^2+bx+c)} = \frac{b}{2c^2} \log\left(\frac{ax^2+bx+c}{x^2}\right) - \frac{1}{cx} + \left(\frac{b^2}{2c^2} - \frac{a}{c}\right) \int \frac{dx}{(ax^2+bx+c)}.$$

119. $$\int \frac{dx}{x^m(ax^2+bx+c)^{n+1}} = -\frac{1}{(m-1)cx^{m-1}(ax^2+bx+c)^n} - \frac{(n+m-1)}{m-1} \frac{b}{c} \int \frac{dx}{x^{m-1}(ax^2+bx+c)^{n+1}} - \frac{(2n+m-1)}{m-1} \frac{a}{c} \int \frac{dx}{x^{m-2}(ax^2+bx+c)^{n+1}}.$$

120. $$\int \frac{dx}{x(ax^2+bx+c)^n} = \frac{1}{2c(n-1)(ax^2+bx+c)^{n-1}} - \frac{b}{2c}\int \frac{dx}{(ax^2+bx+c)^n} + \frac{1}{c}\int \frac{dx}{x(ax^2+bx+c)^{n-1}}.$$

121. $$\int \frac{dx}{ax^2+bx+c} = \frac{1}{\sqrt{a}} \log(2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}), \qquad a>0.$$

122. $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{-a}} \sin^{-1}\frac{-2ax-b}{\sqrt{b^2-4ac}}, \qquad a<0.$$

123. $$\int \frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int \frac{dx}{\sqrt{ax^2+bx+c}}.$$

124. $$\int \frac{x^n\,dx}{\sqrt{ax^2+bx+c}} = \frac{x^{n-1}}{an}\sqrt{ax^2+bx+c} - \frac{b(2n-1)}{2an}\int \frac{x^{n-1}\,dx}{\sqrt{ax^2+bx+c}} - \frac{c(n-1)}{an}\int \frac{x^{n-2}\,dx}{\sqrt{ax^2+bx+c}}.$$

125. $$\int \sqrt{ax^2+bx+c}\,dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \frac{4ac-b^2}{8a}\int \frac{dx}{\sqrt{ax^2+bx+c}}.$$

126. $$\int x\sqrt{ax^2+bx+c}\,dx = \frac{(ax^2+bx+c)^{3/2}}{3a} - \frac{b}{2a}\int \sqrt{ax^2+bx+c}\,dx.$$

127. $$\int x^2\sqrt{ax^2+bx+c}\,dx = \left(x-\frac{5b}{6a}\right)\frac{(ax^2+bx+c)^{3/2}}{4a} + \frac{(5b^2-4ac)}{16a^2}\int \sqrt{ax^2+bx+c}\,dx.$$

128. $$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = -\frac{1}{\sqrt{c}} \log\left(\frac{\sqrt{ax^2+bx+c}+\sqrt{c}}{x} + \frac{b}{2\sqrt{c}}\right), \qquad c>0.$$

129. $$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{-c}} \sin^{-1}\frac{bx+2c}{x\sqrt{b^2-4ac}}, \qquad c<0.$$

130. $$\int \frac{dx}{x\sqrt{ax^2+bx}} = -\frac{2}{bx}\sqrt{ax^2+bx}, \qquad c=0.$$

131. $$\int \frac{dx}{x^n\sqrt{ax^2+bx+c}} = -\frac{\sqrt{ax^2+bx+c}}{c(n-1)x^{n-1}} + \frac{b(3-2n)}{2c(n-1)}\int \frac{dx}{x^{n-1}\sqrt{ax^2+bx+c}} + \frac{a(2-n)}{c(n-1)}\int \frac{dx}{x^{n-2}\sqrt{ax^2+bx+c}}.$$

132. $$\int \frac{dx}{(ax^2+bx+c)^{3/2}} = -\frac{2(2ax+b)}{(b^2-4ac)\sqrt{ax^2+bx+c}}, \qquad b^2 \neq 4ac.$$

133. $\displaystyle\int \frac{dx}{(ax^2+bx+c)^{3/2}} = -\frac{1}{2\sqrt{a^3}(x+b/2a)^2}, \qquad b^2 = 4ac.$

Miscellaneous Algebraic Expression

134. $\displaystyle\int \sqrt{2px-x^2}\,dx = \frac{1}{2}\left[(x-p)\sqrt{2px-x^2} + p^2 \sin^{-1}\left(\frac{x-p}{p}\right)\right].$

135. $\displaystyle\int \frac{dx}{\sqrt{2px-x^2}} = \cos^{-1}\left(\frac{p-x}{p}\right).$

136. $\displaystyle\int \frac{dx}{\sqrt{ax+b}\sqrt{cx+d}} =$

$$\frac{2}{\sqrt{-ac}} \tan^{-1}\sqrt{\frac{-c(ax+b)}{a(cx+d)}} \quad \text{or} \quad \frac{2}{\sqrt{ac}} \tanh^{-1}\sqrt{\frac{c(ax+b)}{a(cx+d)}}.$$

137. $\displaystyle\int \sqrt{ax+b}\,\sqrt{cx+d}\,dx = \frac{(2acx+bc+ad)\sqrt{ax+b}\,\sqrt{cx+d}}{4ac}$

$$+ \frac{(ad-bc)^2}{8ac}\int \frac{dx}{\sqrt{ax+b}\,\sqrt{cx+d}}.$$

138. $\displaystyle\int \sqrt{\frac{cx+d}{ax+b}}\,dx = \frac{\sqrt{ax+b}\,\sqrt{cx+d}}{a} + \frac{(ad-bc)}{2a}\int \frac{dx}{\sqrt{ax+b}\,\sqrt{cx+d}}.$

139. $\displaystyle\int \sqrt{\frac{x+b}{x+d}}\,dx = \sqrt{x+d}\,\sqrt{x+b} + (b-d)\log(\sqrt{x+d}+\sqrt{x+b}).$

140. $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\,dx = \sin^{-1}x - \sqrt{1-x^2}.$

141. $\displaystyle\int \sqrt{\frac{p-x}{q+x}}\,dx = \sqrt{p-x}\,\sqrt{q+x} + (p+q)\sin^{-1}\sqrt{\frac{x+q}{p+q}}.$

142. $\displaystyle\int \sqrt{\frac{p+x}{q-x}}\,dx = -\sqrt{p+x}\,\sqrt{q-x} + (p+q)\sin^{-1}\sqrt{\frac{q-x}{p+q}}.$

143. $\displaystyle\int \frac{dx}{\sqrt{x-p}\,\sqrt{q-x}} = 2\sin^{-1}\sqrt{\frac{x-p}{q-p}}.$

Expressions Containing sin *ax*

144. $\displaystyle\int \sin u\,du = -\cos u,$ where u is any function of x.

145. $\displaystyle\int \sin ax\,dx = -\frac{1}{a}\cos ax.$

146. $\displaystyle\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}.$

147. $\int \sin^3 ax\, dx = -\frac{1}{a}\cos ax + \frac{1}{3a}\cos^3 ax.$

148. $\int \sin^4 ax\, dx = \frac{3x}{8} - \frac{3\sin 2ax}{16a} - \frac{\sin^3 ax \cos ax}{4a}.$

149. $\int \sin^n ax\, dx = -\frac{\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n}\int \sin^{n-2} ax\, dx$ (n pos. integer).

150. $\int \frac{dx}{\sin ax} = \frac{1}{a}\log\left(\tan\frac{ax}{2}\right) = \frac{1}{a}\log(\csc ax - \operatorname{ctn} ax).$

151. $\int \frac{dx}{\sin^2 ax} = \int \csc^2 ax\, dx = -\frac{1}{a}\operatorname{ctn} ax.$

152. $\int \frac{dx}{\sin^n ax} = -\frac{1}{a(n-1)}\frac{\cos ax}{\sin^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} ax}, \quad n \text{ integer} > 1.$

153. $\int \frac{dx}{1 \pm \sin ax} = \mp\frac{1}{a}\tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right).$

154. $\int \frac{dx}{b + c\sin ax} = -\frac{2}{a\sqrt{b^2 - c^2}}\tan^{-1}\left[\sqrt{\frac{b-c}{b+c}}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right)\right], \quad b^2 > c^2.$

155. $\int \frac{dx}{b + c\sin ax} =$

$$\frac{-1}{a\sqrt{c^2 - b^2}}\log\left(\frac{c + b\sin ax + \sqrt{c^2 - b^2}\cos ax}{b + c\sin ax}\right), \quad c^2 > b^2.$$

156. $\int \sin ax \sin bx\, dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}, \quad a^2 \neq b^2.$

157. $\int 1 + \sin x\, dx = \pm 2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)$; use + sign when

$$(8k-1)\frac{\pi}{2} < x \leq (8k+3)\frac{\pi}{2}, \text{ otherwise } -, \ k \text{ an integer.}$$

158. $\int \sqrt{1 - \sin x}\, dx = \pm 2(\sin\frac{x}{2} + \cos\frac{x}{2})$; use + sign when

$$(8k-3)\frac{\pi}{2} < x \leq (8k+1)\frac{\pi}{2}, \text{ otherwise } -, \ k \text{ an integer.}$$

Expressions Involving cos *ax*

159. $\int \cos u\, du = \sin u,$ where u is any function of x.

160. $\int \cos ax\, dx = \frac{1}{a}\sin ax.$

161. $\int \cos^2 ax\, dx = \frac{x}{2} + \frac{\sin 2ax}{4a}.$

162. $\int \cos^3 ax\,dx = \frac{1}{a}\sin ax - \frac{1}{3a}\sin^3 ax.$

163. $\int \cos^4 ax\,dx = \frac{3x}{8} + \frac{3\sin 2ax}{16a} + \frac{\cos^3 ax \sin ax}{4a}.$

164. $\int \cos^n ax\,dx = \frac{\cos^{n-1} ax \sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2} ax\,dx \quad (n \text{ pos. integer}).$

165. $\int \frac{dx}{\cos ax} = \frac{1}{a}\log\left(\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)\right) = \frac{1}{a}\log(\tan ax + \sec ax).$

166. $\int \frac{dx}{\cos^2 ax} = \frac{1}{a}\tan ax.$

167. $\int \frac{dx}{\cos^n ax} = \frac{1}{a(n-1)}\,\frac{\sin ax}{\cos^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} ax}, \quad n \text{ integer} > 1.$

168. $\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2}.$

169. $\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\operatorname{ctn}\frac{ax}{2}.$

170. $\int \sqrt{1+\cos x}\,dx = \pm\sqrt{2}\int \cos\frac{x}{2}\,dx = \pm 2\sqrt{2}\sin\frac{x}{2}$; use + when $(4k-1)\pi < x \leq (4k+1)\pi$, otherwise −, k an integer.

171. $\int \sqrt{1-\cos x}\,dx = \pm\sqrt{2}\int \sin\frac{x}{2}\,dx = \mp 2\sqrt{2}\cos\frac{x}{2}$; use top signs when $4k\pi < x \leq (4k+2)\pi$, otherwise bottom signs.

172. $\int \frac{dx}{b + c\cos ax} = \frac{1}{a\sqrt{b^2 - c^2}}\tan^{-1}\left(\frac{\sqrt{b^2 - c^2}\sin ax}{c + b\cos ax}\right), \qquad b^2 > c^2.$

173. $\int \frac{dx}{b + c\cos ax} = \frac{1}{a\sqrt{c^2 - b^2}}\tanh^{-1}\left(\frac{\sqrt{c^2 - b^2}\sin ax}{c + b\cos ax}\right), \qquad c^2 > b^2.$

174. $\int \cos ax \cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}, \qquad a^2 \neq b^2.$

Expressions Containing sin *ax* and cos *ax*

175. $\int \sin ax \cos bx\,dx = -\frac{1}{2}\left(\frac{\cos(a-b)x}{a-b} + \frac{\cos(a+b)x}{a+b}\right), \qquad a^2 \neq b^2.$

176. $\int \sin^n ax \cos ax\,dx = \frac{1}{a(n+1)}\sin^{n+1} ax, \qquad n \neq -1.$

177. $\int \cos^n ax \sin ax\,dx = -\frac{1}{a(n+1)}\cos^{n+1} ax, \qquad n \neq -1.$

178. $\int \frac{\sin ax}{\cos ax}\,dx = -\frac{1}{a}\log(\cos ax).$

179. $\int \frac{\cos ax}{\sin ax}\,dx = \frac{1}{a}\log(\sin ax).$

180. $\int (b + c\sin ax)^n \cos ax\,dx = \frac{1}{ac(n+1)}(b + c\sin ax)^{n+1}, \qquad n \neq -1.$

181. $\int (b + c\cos ax)^n \sin ax\,dx = -\frac{1}{ac(n+1)}(b + c\cos ax)^{n+1}, \qquad n \neq -1.$

182. $\int \frac{\cos ax\,dx}{b + c\sin ax} = \frac{1}{ac}\ln|b + c\sin a|x.$

183. $\int \frac{\sin ax}{b + c\cos ax}\,dx = -\frac{1}{ac}\log(b + c\cos ax).$

184. $\int \frac{dx}{b\sin ax + c\cos ax} = \frac{1}{a\sqrt{b^2 + c^2}}\left[\log\left(\tan\frac{1}{2}\left(ax + \tan^{-1}\frac{c}{b}\right)\right)\right].$

185. $\int \frac{dx}{b + c\cos ax + d\sin ax} = \frac{-1}{a\sqrt{b^2 - c^2 - d^2}}\sin^{-1} U,$

$$U \equiv \frac{c^2 + d^2 + b(c\cos ax + a\sin ax)}{\sqrt{c^2 + d^2}(b + c\cos ax + d\sin ax)} \quad \text{or} \quad \frac{1}{a\sqrt{c^2 + d^2 - b^2}}\log(V),$$

$$V \equiv \frac{c^2 + d^2 + b(c\cos ax + d\sin ax) + \sqrt{c^2 + d^2 - b^2}(c\sin ax - d\cos ax)}{\sqrt{c^2 + d^2}(b + c\cos ax + d\sin ax)},$$

$$b^2 \neq c^2 + d^2, \quad -\pi < ax < \pi.$$

186. $\int \frac{dx}{b + c\cos ax + d\sin ax} =$

$$\frac{1}{ab}\left(\frac{b - (c+d)\cos ax + (c-d)\sin ax}{b + (c-d)\cos ax + (c+d)\sin ax}\right), \qquad b^2 = c^2 + d^2.$$

187. $\int \frac{\sin^2 ax\,dx}{b + c\cos^2 ax} = \frac{1}{ac}\sqrt{\frac{b+c}{b}}\tan^{-1}\left(\sqrt{\frac{b}{b+c}}\tan ax\right) - \frac{x}{c}.$

188. $\int \frac{\sin ax\cos ax\,dx}{b\cos^2 ax + c\sin^2 ax} = \frac{1}{2a(c-b)}\log(b\cos^2 ax + c\sin^2 ax).$

189. $\int \frac{dx}{b^2\cos^2 ax - c^2\sin^2 ax} = \frac{1}{2abc}\log\left(\frac{b\cos ax + c\sin ax}{b\cos ax - c\sin ax}\right).$

190. $\int \frac{dx}{b^2\cos^2 ax - c^2\sin^2 ax} = \frac{1}{abc}\tan^{-1}\left(\frac{c\tan ax}{b}\right).$

191. $\int \sin^2 ax\cos^2 ax\,dx = \frac{x}{8} - \frac{\sin 4ax}{32a}.$

192. $\int \frac{dx}{\sin ax\cos ax} = \frac{1}{a}\log(\tan ax).$

193. $\int \frac{dx}{\sin^2 ax\cos^2 ax} = \frac{1}{a}(\tan ax - \operatorname{ctn} ax).$

194. $\int \frac{\sin^2 ax}{\cos ax}\,dx = \frac{1}{a}\left[-\sin ax + \log\left(\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)\right)\right].$

195. $\int \frac{\cos^2 ax}{\sin ax}\,dx = \frac{1}{a}\left[\cos ax + \log\left(\tan\frac{ax}{2}\right)\right].$

196. $\int \sin^m ax \cos^n ax\,dx = -\frac{\sin^{m-1} ax \cos^{n+1} ax}{a(m+n)} + \frac{m-1}{m+n}\int \sin^{m-2} ax \cos^n ax\,dx, \qquad m, n > 0.$

197. $\int \sin^m ax \cos^n ax\,dx = \frac{\sin^{m+1} ax \cos^{n-1} ax}{a(m+n)} + \frac{n-1}{m+n}\int \sin^m ax \cos^{n-2} ax\,dx, \qquad m, n > 0.$

198. $\int \frac{\sin^m ax}{\cos^n ax}\,dx = \frac{\sin^{m+1} ax}{a(n-1)\cos^{n-1} ax} - \frac{m-n+2}{n-1}\int \frac{\sin^m ax}{\cos^{n-2} ax}\,dx, \qquad m, n > 0,\ n \neq 1.$

199. $\int \frac{\cos^n ax}{\sin^m ax}\,dx = \frac{-\cos^{n+1} ax}{a(m-1)\sin^{m-1} ax} + \frac{m-n-2}{(m-1)}\int \frac{\cos^n ax}{\sin^{m-2} ax}\,dx, \qquad m, n, > 0,\ m \neq 1.$

200. $\int \frac{dx}{\sin^m ax \cos^n ax} = \frac{1}{a(n-1)}\,\frac{1}{\sin^{m-1} ax \cos^{n-1} ax} + \frac{m+n-2}{(n-1)}\int \frac{dx}{\sin^m ax \cos^{n-2} ax}.$

201. $\int \frac{dx}{\sin^m ax \cos^n ax} = -\frac{1}{a(m-1)}\,\frac{1}{\sin^{m-1} ax \cos^{n-1} ax} + \frac{m+n-2}{(m-1)}\int \frac{dx}{\sin^{m-2} ax \cos^n ax}.$

202. $\int \frac{\sin^{2n} ax}{\cos ax}\,dx = \int \frac{(1-\cos^2 ax)^n}{\cos ax}\,dx.$ (Expand, divide, and use 205).

203. $\int \frac{\cos^{2n} ax}{\sin ax}\,dx = \int \frac{(1-\sin^2 ax)^n}{\sin ax}\,dx.$ (Expand, divide, and use 190).

204. $\int \frac{\sin^{2n+1} ax}{\cos ax}\,dx = \int \frac{(1-\cos^2 ax)^n}{\cos ax}\sin ax\,dx.$ (Expand, divide, and use 218).

205. $\int \frac{\cos^{2n+1} ax}{\sin ax}\,dx = \int \frac{(1-\sin^2 ax)^n}{\sin ax}\cos ax\,dx.$ (Expand, divide, and use 217).

Expressions Containing tan *ax* or ctn *ax* (tan *ax* = 1/ctn *ax*)

206. $\int \tan u\,du = -\log(\cos u)$ or $\log(\sec u)$, where u is any function of x.

207. $\int \tan ax\,dx = -\frac{1}{a}\log(\cos ax)$.

208. $\int \tan^2 ax\,dx = \frac{1}{a}\tan ax - x$.

209. $\int \tan^3 ax\,dx = \frac{1}{2a}\tan^2 ax + \frac{1}{a}\log(\cos ax)$.

210. $\int \tan^n ax\,dx = \frac{1}{a(n-1)}\tan^{n-1} ax - \int \tan^{n-2} ax\,dx, \qquad n \text{ integer} > 1.$

211. $\int \operatorname{ctn} u\,du = \log(\sin u)$ or $-\log(\csc u)$, where u is any function of x.

212. $\int \operatorname{ctn}^2 ax\,dx = \int \frac{dx}{\tan^2 ax} = -\frac{1}{a}\operatorname{ctn} ax - x$.

213. $\int \operatorname{ctn}^3 ax\,dx = -\frac{1}{2a}\operatorname{ctn}^2 ax - \frac{1}{a}\log(\sin ax)$.

214. $\int \operatorname{ctn}^n ax\,dx = \int \frac{dx}{\tan^n ax}$

$$-\frac{1}{a(n-1)}\operatorname{ctn}^{n-1} ax - \int \operatorname{ctn}^{n-2} ax\,dx, \; n \text{ integer} > 1.$$

215. $\int \frac{dx}{b + c\tan ax} = \int \frac{\operatorname{ctn} ax\,dx}{b\operatorname{ctn} ax + c}$

$$= \frac{1}{b^2 + c^2}\left[bx + \frac{c}{a}\log(b\cos ax + c\sin ax)\right].$$

216. $\int \frac{dx}{b + c\operatorname{ctn} ax} = \int \frac{\tan ax\,dx}{b\tan ax + c}$

$$= \frac{1}{b^2 + c^2}\left(bx - \frac{c}{a}\log(c\cos ax + b\sin ax)\right).$$

217. $\int \frac{dx}{\sqrt{b + c\tan^2 ax}} = \frac{1}{a\sqrt{b-c}}\sin^{-1}\left(\sqrt{\frac{b-c}{b}}\sin ax\right), \qquad b \text{ pos.}, \; b^2 > c^2.$

Expressions Containing sec *ax* = 1/cos *ax* or csc *ax* = 1/sin *ax*

218. $\int \sec u\, du = \log(\sec u + \tan u) = \log\left[\tan\left(\frac{u}{2} + \frac{\pi}{4}\right)\right]$,
where u is any function of x.

219. $\int \sec ax\, dx = \frac{1}{a}\log\left[\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)\right]$.

220. $\int \sec^2 ax\, dx = \frac{1}{a}\tan ax$.

221. $\int \sec^3 ax\, dx = \frac{1}{2a}\left[\tan ax \sec ax + \log\left(\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)\right)\right]$.

222. $\int \sec^n ax\, dx = \frac{1}{a(n-1)}\frac{\sin ax}{\cos^{n-1} ax} + \frac{n-2}{n-1}\int \sec^{n-2} ax\, dx, \quad n \text{ integer} > 1.$

223. $\int \csc u\, du = \log(\csc u - \operatorname{ctn} u) = \log\left(\tan\frac{u}{2}\right)$,
where u is any function of x.

224. $\int \csc ax\, dx = \frac{1}{a}\log\left(\tan\frac{ax}{2}\right)$.

225. $\int \csc^2 ax\, dx = -\frac{1}{a}\operatorname{ctn} ax$.

226. $\int \csc^3 ax\, dx = \frac{1}{2a}\left[-\operatorname{ctn} ax \csc ax + \log\left(\tan\frac{ax}{2}\right)\right]$.

227. $\int \csc^n ax\, dx = -\frac{1}{a(n-1)}\frac{\cos ax}{\sin^{n-1} ax} + \frac{n-2}{n-1}\int \csc^{n-2} ax\, dx, \quad n \text{ integer} > 1.$

Expressions Containing tan *ax* and sec *ax* or ctn *ax* and csc *ax*

228. $\int \tan u \sec u\, du = \sec u$, where u is any function of x.

229. $\int \tan ax \sec ax\, dx = \frac{1}{a}\sec ax$.

230. $\int \tan^n ax \sec^2 ax\, dx = \frac{1}{a(n+1)}\tan^{n+1} ax, \qquad n \neq -1.$

231. $\int \tan ax \sec^n ax\, dx = \frac{1}{an} \sec^n ax, \qquad n \neq 0.$

232. $\int \operatorname{ctn} u \csc u\, du = -\csc u, \qquad$ where u is any function of x.

233. $\int \operatorname{ctn} ax \csc ax\, dx = -\frac{1}{a} \csc ax.$

234. $\int \operatorname{ctn}^n ax \csc^2 ax\, dx = -\frac{1}{a(n+1)} \operatorname{ctn}^{n+1} ax, \qquad n \neq -1.$

235. $\int \operatorname{ctn} ax \csc^n ax\, dx = -\frac{1}{an} \csc^n ax, \qquad n \neq 0.$

236. $\int \frac{\csc^2 ax\, dx}{\operatorname{ctn} ax} = -\frac{1}{a} \log(\operatorname{ctn} ax).$

Expressions Containing Algebraic and Trigonometric Functions

237. $\int x \sin ax\, dx = \frac{1}{a^2} \sin ax - \frac{1}{a} x \cos ax.$

238. $\int x^2 \sin ax\, dx = \frac{2x}{a^2} \sin ax + \frac{2}{a^3} \cos ax - \frac{x^2}{a} \cos ax.$

239. $\int x^3 \sin ax\, dx = \frac{3x^2}{a^2} \sin ax - \frac{6}{a^4} \sin ax - \frac{x^3}{a} \cos ax + \frac{6x}{a^3} \cos ax.$

240. $\int x \sin^2 ax\, dx = \frac{x^2}{4} - \frac{x \sin 2ax}{4a} - \frac{\cos 2ax}{8a^2}.$

241. $\int x^2 \sin^2 ax\, dx = \frac{x^3}{6} - \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right) \sin 2ax - \frac{x \cos 2ax}{4a^2}.$

242. $\int x^3 \sin^2 ax\, dx = \frac{x^4}{8} - \left(\frac{x^3}{4a} - \frac{3x}{8a^3}\right) \sin 2ax - \left(\frac{3x^2}{8a^2} - \frac{3}{16a^4}\right) \cos 2ax.$

243. $\int x \sin^3 ax\, dx = \frac{x \cos 3ax}{12a} - \frac{\sin 3ax}{36a^2} - \frac{3x \cos ax}{4a} + \frac{3 \sin ax}{4a^2}.$

244. $\int x^n \sin ax\, dx = -\frac{1}{a} x^n \cos ax + \frac{n}{a} \int x^{n-1} \cos ax\, dx, \qquad n > 0.$

245. $\int \frac{\sin ax\, dx}{x} = ax - \frac{(ax)^3}{3 \cdot 3!} + \frac{(ax)^5}{5 \cdot 5!} - \cdots.$

246. $\int \frac{\sin ax\, dx}{x^m} = \frac{-1}{(m-1)} \frac{\sin ax}{x^{m-1}} + \frac{a}{(m-1)} \int \frac{\cos ax\, dx}{x^{m-1}}.$

247. $\int x \cos ax\, dx = \frac{1}{a^2} \cos ax + \frac{1}{a} x \sin ax.$

248. $\int x^2 \cos ax\, dx = \frac{2x}{a^2}\cos ax - \frac{2}{a^3}\sin ax + \frac{x^2}{a}\sin ax.$

249. $\int x^3 \cos ax\, dx = \frac{(3a^2x^2 - 6)\cos ax}{a^4} + \frac{(a^2x^3 - 6x)\sin a}{a^3}.$

250. $\int x\cos^2 ax\, dx = \frac{x^2}{4} + \frac{x\sin 2ax}{4a} + \frac{\cos 2ax}{8a^2}.$

251. $\int x^2\cos^2 ax\, dx = \frac{x^3}{6} + \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin 2ax + \frac{x\cos 2ax}{4a^2}.$

252. $\int x^3\cos^2 ax\, dx = \frac{x^4}{8} + \left(\frac{x^3}{4a} - \frac{3x}{8a^3}\right)\sin 2ax + \left(\frac{3x^2}{8a^2} - \frac{3}{16a^4}\right)\cos 2ax.$

253. $\int x\cos^3 ax\, dx = \frac{x\sin 3ax}{12a} + \frac{\cos 3ax}{36a^2} + \frac{3x\sin ax}{4a} + \frac{3\cos ax}{4a^2}.$

254. $\int x^n \cos ax\, dx = \frac{1}{a}x^n \sin ax - \frac{n}{a}\int x^{n-1}\sin ax\, dx, \quad n \text{ pos.}$

255. $\int \frac{\cos ax\, dx}{x} = \log(ax) - \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^4}{4\cdot 4!} - \cdots.$

256. $\int \frac{\cos ax}{x^m}\, dx = -\frac{1}{(m-1)}\frac{\cos ax}{x^{m-1}} - \frac{a}{(m-1)}\int \frac{\sin ax\, dx}{x^{m-1}}.$

Expressions Containing Exponential and Logarithmic Functions

257. $\int e^u\, du = e^u,$ where u is any function of x.

258. $\int b^u\, du = \frac{b^u}{\log(b)},$ where u is any function of x.

259. $\int e^{ax}\, dx = \frac{1}{a}e^{ax}, \qquad \int b^{ax}\, dx = \frac{b^{ax}}{a\log(b)}.$

260. $\int xe^{ax}\, dx = \frac{e^{ax}}{a^2}(ax - 1), \qquad \int xb^{ax}\, dx = \frac{xb^{ax}}{a\log(b)} - \frac{b^{ax}}{a^2(\log(b))^2}.$

261. $\int x^2e^{ax}\, dx = \frac{e^{ax}}{a^3}(a^2x^2 - 2ax + 2).$

262. $\int x^n e^{ax}\, dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\, dx, \quad n \text{ pos.}$

263. $\int x^n e^{ax}\, dx = \frac{e^{ax}}{a^{n+1}}[(ax)^n - n(ax)^{n-1} + n(n-1)(ax)^{n-2} - \cdots + (-1)^n n!], \quad n \text{ pos. integ.}$

264. $\int x^n e^{-ax}\,dx = -\frac{e^{-ax}}{a^{n+1}}[(ax)^n + n(ax)^{n-1} + n(n-1)(ax)^{n-2} + \cdots + n!], \quad n \text{ pos. integ.}$

265. $\int x^n b^{ax}\,dx = \frac{x^n b^{ax}}{a\log(b)} - \frac{n}{a\log(b)}\int x^{n-1} b^{ax}\,dx, \quad n \text{ pos.}$

266. $\int \frac{e^{ax}}{x}\,dx = \log(x) + ax + \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^3}{3\cdot 3!} + \cdots.$

267. $\int \frac{e^{ax}}{x^n}\,dx = \frac{1}{n-1}\left(-\frac{e^{ax}}{x^{n-1}} + a\int \frac{e^{ax}}{x^{n-1}}\,dx\right), \quad n \text{ integ.} > 1.$

268. $\int \frac{dx}{b + ce^{ax}} = \frac{1}{ab}[ax - \log(b + ce^{ax})].$

269. $\int \frac{e^{ax}\,dx}{b + ce^{ax}} = \frac{1}{ac}\log(b + ce^{ax}).$

270. $\int \frac{dx}{be^{ax} + ce^{-ax}} = \frac{1}{a\sqrt{bc}}\tan^{-1}\left(e^{ax}\sqrt{\frac{b}{c}}\right), \quad b \text{ and } c \text{ pos.}$

271. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx).$

272. $\int e^{ax}\sin bx\sin cx\,dx = \frac{e^{ax}[(b-c)\sin(b-c)x + a\cos(b-c)x]}{2[a^2 + (b-c)^2]} - \frac{e^{ax}[(b+c)\sin(b+c) + a\cos(b+c)x]}{2[a^2 + (b+c)^2]}.$

273. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx).$

274. $\int e^{ax}\cos bx\cos cx\,dx = \frac{e^{ax}[(b-c)\sin(b-c)x + a\cos(b-c)x]}{2[a^2 + (b-c)^2]} + \frac{e^{ax}[(b+c)\sin(b+c)x + a\cos(b+c)x]}{2[a^2 + (b+c)^2]}.$

275. $\int e^{ax}\sin bx\cos cx\,dx = \frac{e^{ax}[a\sin(b-c)x - (b-c)\cos(b-c)x]}{2[a^2 + (b-c)^2]} + \frac{e^{ax}[a\sin(b+c)x - (b+c)\cos(b+c)x]}{2[a^2 + (b+c)^2]}.$

276. $\int e^{ax}\sin bx\sin(bx + c)\,dx = \frac{e^{ax}\cos c}{2a} - \frac{e^{ax}[a\cos(2bx + c) + 2b\sin(2bx + c)]}{2(a^2 + 4b^2)}.$

277. $\int e^{ax}\cos bx\cos(bx + c)\,dx = \frac{e^{ax}\cos c}{2a} + \frac{e^{ax}[a\cos(2bx + c) + 2b\sin(2bx + c)]}{2(a^2 + 4b^2)}.$

278. $\int e^{ax} \sin bx \cos(bx + c)\, dx = -\dfrac{e^{ax} \sin c}{2a} + \dfrac{e^{ax}[a \sin(2bx + c) - 2b \cos(2bx + c)]}{2(a^2 + 4b^2)}.$

279. $\int e^{ax} \cos bx \sin(bx + c)\, dx = \dfrac{e^{ax} \sin c}{2a} + \dfrac{e^{ax}[a \sin(2bx + c) - 2b \cos(2bx + c)]}{2(a^2 + 4b^2)}.$

280. $\int xe^{ax} \sin bx\, dx = \dfrac{xe^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) - \dfrac{e^{ax}}{(a^2 + b^2)^2}[(a^2 - b^2) \sin bx - 2ab \cos bx].$

281. $\int xe^{ax} \cos bx\, dx = \dfrac{xe^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) - \dfrac{e^{ax}}{(a^2 + b^2)^2}[(a^2 - b^2) \cos bx + 2ab \sin bx].$

282. $\int e^{ax} \cos^n bx\, dx = \dfrac{e^{ax} (\cos^{n-1} bx)(a \cos bx + nb \sin bx)}{a^2 + n^2 b^2} + \dfrac{n(n-1)b^2}{a^2 + n^2 b^2} \int e^{ax} \cos^{n-2} bx\, dx.$

283. $\int e^{ax} \sin^n bx\, dx = \dfrac{e^{ax} (\sin^{n-1} bx)(a \sin bx - nb \cos bx)}{a^2 + n^2 b^2} + \dfrac{n(n-1)b^2}{a^2 + n^2 b^2} \int e^{ax} \sin^{n-2} bx\, dx.$

284. $\int \log(ax)\, dx = x \log(ax) - x.$

285. $\int x \log(ax)\, dx = \dfrac{x^2}{2} \log(ax) - \dfrac{x^2}{4}.$

286. $\int x^2 \log(ax)\, dx = \dfrac{x^3}{3} \log(ax) - \dfrac{x^3}{9}.$

287. $\int [\log(ax)]^2\, dx = x[\log(ax)]^2 - 2x \log(ax) + 2x.$

288. $\int [\log(ax)]^n\, dx = x[\log(ax)]^n - n \int [\log(ax)]^{n-1}\, dx, \quad n \text{ pos.}$

289. $\int x^n \log(ax)\, dx = x^{n+1}\left(\dfrac{\log(ax)}{n+1} - \dfrac{1}{(n+1)^2}\right), \qquad n \neq -1.$

290. $\int x^n [\log(ax)]^m\, dx = \dfrac{x^{n+1}}{n+1}[\log(ax)]^m - \dfrac{m}{n+1} \int x^n [\log(ax)]^{m-1}\, dx.$

291. $\int \frac{[\log(ax)]^n}{x}\,dx = \frac{[\log(ax)]^{n+1}}{n+1}, \qquad n \neq -1.$

292. $\int \frac{dx}{x\log(ax)} = \log[\log(ax)].$

293. $\int \frac{dx}{x(\log(ax))^n} = -\frac{1}{(n-1)(\log(ax))^{n-1}}.$

294. $\int \frac{x^n\,dx}{(\log(ax))^m} =$

$$\frac{-x^{n+1}}{(m-1)(\log(ax))^{m-1}} + \frac{n+1}{m-1}\int \frac{x^n\,dx}{(\log(ax))^{m-1}}, \qquad m \neq 1.$$

295. $\int \frac{x^n\,dx}{\log(ax)} = \frac{1}{a^{n+1}}\int \frac{e^y\,dy}{y}, \qquad y = (n+1)\ln|ax|.$

296. $\int \frac{x^n\,dx}{\log(ax)} = \frac{1}{a^{n+1}}[\log|\log(ax)| + (n+1)\log(ax)$

$$+ \frac{(n+1)^2[\log(ax)]^2}{2\cdot 2!} + \frac{(n+1)^3[\log(ax)]^3}{3\cdot 3!} + \cdots].$$

297. $\int \frac{dx}{\log(ax)} =$

$$\frac{1}{a}[\log[\log(ax)] + \log(ax) + \frac{(\log(ax))^2}{2\cdot 2!} + \frac{(\log(ax))^3}{3\cdot 3!} + \cdots].$$

298. $\int \sin[\log(ax)]\,dx = \frac{x}{2}[\sin[\log(ax)] - \cos[\log(ax)].$

299. $\int \cos[\log(ax)]\,dx = \frac{x}{2}[\sin[\log(ax)] + \cos[\log(ax)].$

300. $\int e^{ax}\log(bx)\,dx = \frac{1}{a}e^{ax}\log(bx) - \frac{1}{a}\int \frac{e^{ax}}{x}\,dx.$

Expressions Containing Inverse Trigonometric Functions

301. $\int \sin^{-1}ax\,dx = x\sin^{-1}ax + \frac{1}{a}\sqrt{1-a^2x^2}.$

302. $\int (\sin^{-1}ax)^2\,dx = x(\sin^{-1}ax)^2 - 2x + \frac{2}{a}\sqrt{1-a^2x^2}\sin^{-1}ax.$

303. $\int x\sin^{-1}ax\,dx = \frac{x^2}{2}\sin^{-1}ax - \frac{1}{4a^2}\sin^{-1}ax + \frac{x}{4a}\sqrt{1-a^2x^2}.$

304. $\int x^n\sin^{-1}ax\,dx = \frac{x^{n+1}}{n+1}\sin^{-1}ax - \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1-a^2x^2}}, \qquad n \neq -1.$

305. $$\int \frac{\sin^{-1} ax\,dx}{x} = ax + \frac{1}{2\cdot 3\cdot 3}(ax)^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 5}(ax)^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 7}(ax)^7 + \cdots, \qquad a^2x^2 < 1.$$

306. $$\int \frac{\sin^{-1} ax\,dx}{x^2} = -\frac{1}{x}\sin^{-1} ax - a\log\left(\frac{1+\sqrt{1-a^2x^2}}{ax}\right).$$

307. $$\int \cos^{-1} ax\,dx = x\cos^{-1} ax - \frac{1}{a}\sqrt{1-a^2x^2}.$$

308. $$\int (\cos^{-1} ax)^2\,dx = x(\cos^{-1} ax)^2 - 2x - \frac{2}{a}\sqrt{1-a^2x^2}\cos^{-1} ax.$$

309. $$\int x\cos^{-1} ax\,dx = \frac{x^2}{2}\cos^{-1} ax - \frac{1}{4a^2}\cos^{-1} ax - \frac{x}{4a}\sqrt{1-a^2x^2}.$$

310. $$\int x^n\cos^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\cos^{-1} ax + \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1-a^2x^2}}, \qquad n \neq -1.$$

311. $$\int \frac{\cos^{-1} ax\,dx}{x} = \frac{\pi}{2}\ln|ax| - ax - \frac{1}{2\cdot 3\cdot 3}(ax)^3 - \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 5}(ax)^5 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 7}(ax)^7 - \cdots, \qquad a^2x^2 < 1.$$

312. $$\int \frac{\cos^{-1} ax\,dx}{x^2} = -\frac{1}{x}\cos^{-1} ax + a\log\left(\frac{1+\sqrt{1-a^2x^2}}{ax}\right).$$

313. $$\int \tan^{-1} ax\,dx = x\tan^{-1} ax - \frac{1}{2a}\log(1+a^2x^2).$$

314. $$\int x^n\tan^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\tan^{-1} ax - \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{1+a^2x^2}, \qquad n \neq -1.$$

315. $$\int \frac{\tan^{-1} ax\,dx}{x^2} = -\frac{1}{x}\tan^{-1} ax - \frac{a}{2}\log\left(\frac{1+a^2x^2}{a^2x^2}\right).$$

316. $$\int \operatorname{ctn}^{-1} ax\,dx = x\operatorname{ctn}^{-1} ax + \frac{1}{2a}\log(1+a^2x^2).$$

317. $$\int x^n\operatorname{ctn}^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\operatorname{ctn}^{-1} ax + \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{1+a^2x^2}, \qquad n \neq -1.$$

318. $$\int \frac{\operatorname{ctn}^{-1} ax\,dx}{x^2} = -\frac{1}{x}\operatorname{ctn}^{-1} ax + \frac{a}{2}\log\left(\frac{1+a^2x^2}{a^2x^2}\right).$$

319. $$\int \sec^{-1} ax\,dx = x\sec^{-1} ax - \frac{1}{a}\log(ax + \sqrt{a^2x^2-1}).$$

320. $$\int x^n\sec^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\sec^{-1} ax \pm \frac{1}{n+1}\int \frac{x^n\,dx}{\sqrt{a^2x^2-1}}, \qquad n \neq -1;$$

use + sign when $\pi/2 < \sec^{-1} ax < \pi$; − sign when $0 < \sec^{-1} ax < \pi/2$.

321. $\int \csc^{-1} ax\, dx = x \csc^{-1} ax + \frac{1}{a} \log(ax + \sqrt{a^2x^2 - 1})$.

322. $\int x^n \csc^{-1} ax\, dx = \frac{x^{n+1}}{n+1} \csc^{-1} ax \pm \frac{1}{n+1} \int \frac{x^n\, dx}{\sqrt{a^2x^2 - 1}}, \qquad n \neq -1;$

use + sign when $0 < \csc^{-1} ax < \pi/2$; − sign when $-\pi/2 < \csc^{-1} ax < 0$.

Index