
Appendix:
TABLE OF INTEGRALS

148. In the following table, the constant of integration, C , is omitted but should be added to the result of every integration. The letter x represents any variable; u represents any function of x ; the remaining letters represent arbitrary constants, unless otherwise indicated; all angles are in radians. Unless otherwise mentioned, $\log_e u = \log u$.

Expressions Containing $ax + b$

$$1. \int (ax + b)^n dx = \frac{1}{a(n+1)}(ax + b)^{n+1}, \quad n \neq -1.$$

$$2. \int \frac{dx}{ax + b} = \frac{1}{a} \log_e(ax + b).$$

$$3. \int \frac{dx}{(ax + b)^2} = -\frac{1}{a(ax + b)}.$$

$$4. \int \frac{dx}{(ax + b)^3} = -\frac{1}{2a(ax + b)^2}.$$

$$5. \int x(ax + b)^n dx = \frac{1}{a^2(n+2)}(ax + b)^{n+2} \\ - \frac{b}{a^2(n+1)}(ax + b)^{n+1}, \quad n \neq -1, -2.$$

$$6. \int \frac{x dx}{ax + b} = \frac{x}{a} - \frac{b}{a^2} \log(ax + b).$$

$$7. \int \frac{x dx}{(ax + b)^2} = \frac{b}{a^2(ax + b)} + \frac{1}{a^2} \log(ax + b).$$

8. $\int \frac{x \, dx}{(ax + b)^3} = \frac{b}{2a^2(ax + b)^2} - \frac{1}{a^2(ax + b)}$.
9. $\int x^2(ax + b)^n \, dx = \frac{1}{a^2} \left(\frac{(ax + b)^{n+3}}{n + 3} - 2b \frac{(ax + b)^{n+2}}{n + 2} + \frac{b^2(ax + b)^{n+1}}{n + 1} \right), \quad n \neq -1, -2, -3$.
10. $\int \frac{x^2 \, dx}{ax + b} = \frac{1}{a^3} \left(\frac{1}{2}(ax + b)^2 - 2b(ax + b) + b^2 \log(ax + b) \right)$.
11. $\int \frac{x^2 \, dx}{(ax + b)^2} = \frac{1}{a^3} \left((ax + b) - 2b \log(ax + b) - \frac{b^2}{ax + b} \right)$.
12. $\int \frac{x^2 \, dx}{(ax + b)^3} = \frac{1}{a^3} \left(\log(ax + b) + \frac{2b}{ax + b} - \frac{b^2}{2(ax + b)^2} \right)$.
13. $\int x^m(ax + b)^n \, dx$
 $= \frac{1}{a(m + n + 1)} \left(x^m(ax + b)^{n+1} - mb \int x^{m-1}(ax + b)^n \, dx \right)$
 $= \frac{1}{m + n + 1} \left(x^{m+1}(ax + b)^n + nb \int x^m(ax + b)^{n-1} \, dx \right),$
 $m > 0, m + n + 1 \neq 0$.
14. $\int \frac{dx}{x(ax + b)} = \frac{1}{b} \log \left(\frac{x}{ax + b} \right)$.
15. $\int \frac{dx}{x^2(ax + b)} = -\frac{1}{bx} + \frac{a}{b^2} \log \left(\frac{ax + b}{x} \right)$.
16. $\int \frac{dx}{x^3(ax + b)} = \frac{2ax - b}{2b^2x^2} + \frac{a^2}{b^3} \log \left(\frac{x}{ax + b} \right)$.
17. $\int \frac{dx}{x(ax + b)^2} = \frac{1}{b(ax + b)} - \frac{1}{b^2} \log \left(\frac{ax + b}{x} \right)$.
18. $\int \frac{dx}{x(ax + b)^3} = \frac{1}{b^3} \left[\frac{1}{2} \left(\frac{ax + 2b}{ax + b} \right)^2 + \log \left(\frac{x}{ax + b} \right) \right]$.
19. $\int \frac{dx}{x^2(ax + b)^2} = -\frac{b + 2ax}{b^2x(ax + b)} + \frac{2a}{b^3} \log \left(\frac{ax + b}{x} \right)$.
20. $\int \sqrt{ax + b} \, dx = \frac{2}{3a} \sqrt{(ax + b)^3}$.
21. $\int x \sqrt{ax + b} \, dx = \frac{2(3ax - 2b)}{15a^2} \sqrt{(ax + b)^3}$.
22. $\int x^2 \sqrt{ax + b} \, dx = \frac{2(15a^2x^2 - 12abx + 8b^2) \sqrt{(ax + b)^3}}{105a^3}$.
23. $\int x^3 \sqrt{ax + b} \, dx = \frac{2(35a^3x^3 - 30a^2bx^2 + 24ab^2x - 16b^3) \sqrt{(ax + b)^3}}{315a^4}$.

24. $\int x^n \sqrt{ax+b} dx = \frac{2}{a^{n+1}} \int u^2 (u^2 - b)^n du, \quad u = \sqrt{ax+b}.$
25. $\int \frac{\sqrt{ax+b}}{x} dx = 2\sqrt{ax+b} + b \int \frac{dx}{x\sqrt{ax+b}}.$
26. $\int \frac{dx}{\sqrt{ax+b}} = \frac{2\sqrt{ax+b}}{a}.$
27. $\int \frac{x dx}{\sqrt{ax+b}} = \frac{2(ax-2b)}{3a^2} \sqrt{ax+b}.$
28. $\int \frac{x^2 dx}{\sqrt{ax+b}} = \frac{2(3a^2x^2 - 4abx + 8b^2)}{15a^3} \sqrt{ax+b}.$
29. $\int \frac{x^3 dx}{\sqrt{ax+b}} = \frac{2(5a^3x^3 - 6a^2bx^2 + 8ab^2x - 16b^3)}{35a^4} \sqrt{ax+b}.$
30. $\int \frac{x^n dx}{\sqrt{ax+b}} = \frac{2}{a^{n+1}} \int (u^2 - b)^n du, \quad u = \sqrt{ax+b}.$
31. $\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}} \log\left(\frac{\sqrt{ax+b} - \sqrt{b}}{\sqrt{ax+b} + \sqrt{b}}\right) \quad \text{for } b > 0.$
32. $\int \frac{dx}{x\sqrt{ax+b}} = \frac{2}{\sqrt{-b}} \tan^{-1} \sqrt{\frac{ax+b}{-b}}, \quad b < 0$
 $= \frac{-2}{\sqrt{b}} \tanh^{-1} \sqrt{\frac{ax+b}{-b}}, \quad b > 0.$
33. $\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b} \int \frac{dx}{x\sqrt{ax+b}}.$
34. $\int \frac{dx}{x^3\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{2bx^2} + \frac{3a\sqrt{ax+b}}{4b^2x} + \frac{3a^2}{8b^2} \int \frac{dx}{x\sqrt{ax+b}}.$
35. $\int \frac{dx}{x^n(ax+b)^m} = -\frac{1}{b^{m+n-1}} \int \frac{(u-a)^{m+n-2} du}{u^m}, \quad u = \frac{ax+b}{x}.$
36. $\int (ax+b)^{\pm n/2} dx = \frac{2(ax+b)^{(2\pm n)/2}}{a(2\pm n)}.$
37. $\int x(ax+b)^{\pm n/2} dx = \frac{2}{a^2} \left(\frac{(ax+b)^{(4\pm n)/2}}{4\pm n} - \frac{b(ax+b)^{(2\pm n)/2}}{2\pm n} \right).$
38. $\int \frac{dx}{x(ax+b)^{n/2}} = \frac{1}{b} \int \frac{dx}{x(ax+b)^{(n-2)/2}} - \frac{a}{b} \int \frac{dx}{(ax+b)^{n/2}}.$
39. $\int \frac{x^m dx}{\sqrt{ax+b}} = \frac{2x^m\sqrt{ax+b}}{(2m+1)a} - \frac{2mb}{(2m+1)a} \int \frac{x^{m-1} dx}{\sqrt{ax+b}}.$
40. $\int \frac{dx}{x^n\sqrt{ax+b}} = \frac{-\sqrt{ax+b}}{(n-1)bx^{n-1}} - \frac{(2n-3)a}{(2n-2)b} \int \frac{dx}{x^{n-1}\sqrt{ax+b}}.$
41. $\int \frac{(ax+b)^{n/2}}{x} dx = a \int (ax+b)^{(n-2)/2} dx + b \int \frac{(ax+b)^{(n-2)/2}}{x} dx.$

42. $\int \frac{dx}{(ax+b)(cx+d)} = \frac{1}{bc-ad} \log\left(\frac{cx+d}{ax+b}\right), \quad bc-ad \neq 0.$
43. $\int \frac{dx}{(ax+b)^2(cx+d)} = \frac{1}{bc-ad} \left[\frac{1}{ax+b} + \frac{c}{bc-ad} \log\left(\frac{cx+d}{ax+b}\right) \right], \quad bc-ad \neq 0.$
44. $\int (ax+b)^n(cx+d)^m dx = \frac{1}{(m+n+1)a} \left[(ax+b)^{n+1}(cx+d)^m - m(bc-ad) \int (ax+b)^n(cx+d)^{m-1} dx \right].$
45. $\int \frac{dx}{(ax+b)^n(cx+d)^m} = \frac{1}{(m-1)(bc-ad)} \left(\frac{1}{(ax+b)^{n-1}(cx+d)^{m-1}} - a(m+n-2) \int \frac{dx}{(ax+b)^n(cx+d)^{m-1}} \right), \quad m > 1, n > 0, bc-ad \neq 0.$
46. $\int \frac{(ax+b)^n}{(cx+d)^m} dx = -\frac{1}{(m-1)(bc-ad)} \left(\frac{(ax+b)^{n+1}}{(cx+d)^{m-1}} + (m-n-2)a \int \frac{(ax+b)^n dx}{(cx+d)^{m-1}} \right) = \frac{1}{(m-n-1)c} \left(\frac{(ax+b)^n}{(cx+d)^{m-1}} + n(bc-ad) \int \frac{(ax+b)^{n-1}}{(cx+d)^m} dx \right).$
47. $\int \frac{x dx}{(ax+b)(cx+d)} = \frac{1}{bc-ad} \left(\frac{b}{a} \ln|ax+b| - \frac{d}{c} \log(cx+d) \right), \quad bc-ad \neq 0.$
48. $\int \frac{x dx}{(ax+b)^2(cx+d)} = \frac{1}{bc-ad} \left(-\frac{b}{a(ax+b)} - \frac{d}{bc-ad} \log\left(\frac{cx+d}{ax+b}\right) \right), \quad bc-ad \neq 0.$
49. $\int \frac{cx+d}{\sqrt{ax+b}} dx = \frac{2}{3a^2} (3ad - 2bc + acx) \sqrt{ax+b}.$
50. $\int \frac{\sqrt{ax+b}}{cx+d} dx = \frac{2\sqrt{ax+b}}{c} - \frac{2}{c} \sqrt{\frac{ad-bc}{c}} \tan^{-1} \sqrt{\frac{c(ax+b)}{ad-bc}}, \quad c > 0, ad > bc.$
51. $\int \frac{\sqrt{ax+b}}{cx+d} dx = \frac{2\sqrt{ax+b}}{c} + \frac{1}{c} \sqrt{\frac{bc-ad}{c}} \log\left(\frac{\sqrt{c(ax+b)} - \sqrt{bc-ad}}{\sqrt{c(ax+b)} + \sqrt{bc-ad}}\right), \quad c > 0, bc > ad.$
52. $\int \frac{dx}{(cx+d)\sqrt{ax+b}} = \frac{2}{\sqrt{c}\sqrt{ad-bc}} \tan^{-1} \sqrt{\frac{c(ax+b)}{ad-bc}}, \quad c > 0, ad > bc.$

$$53. \int \frac{dx}{(cx+d)\sqrt{ax+b}} = \frac{1}{\sqrt{c}\sqrt{bc-ad}} \log\left(\frac{\sqrt{c(ax+b)} - \sqrt{bc-ad}}{\sqrt{c(ax+b)} + \sqrt{bc-ad}}\right), \quad c > 0, bc > ad.$$

Expressions Containing $ax^2 + c$, $ax^n + c$, $x^2 \pm p^2$, and $p^2 - x^2$

$$54. \int \frac{dx}{p^2 + x^2} = \frac{1}{p} \tan^{-1} \frac{x}{p} \quad \text{or} \quad -\frac{1}{p} \operatorname{ctn}^{-1}\left(\frac{x}{p}\right).$$

$$55. \int \frac{dx}{p^2 - x^2} = \frac{1}{2p} \log\left(\frac{p+x}{p-x}\right), \quad \text{or} \quad \frac{1}{p} \tanh^{-1}\left(\frac{x}{p}\right).$$

$$56. \int \frac{dx}{ax^2 + c} = \frac{1}{\sqrt{ac}} \tan^{-1}\left(x\sqrt{\frac{a}{c}}\right), \quad a \text{ and } c > 0.$$

$$57. \int \frac{dx}{ax^2 + c} = \frac{1}{2\sqrt{-ac}} \log\left(\frac{x\sqrt{a} - \sqrt{-c}}{x\sqrt{a} + \sqrt{-c}}\right), \quad n > 0, c < 0$$

$$= \frac{1}{2\sqrt{-ac}} \log\left(\frac{\sqrt{c} + x\sqrt{-a}}{\sqrt{c} - x\sqrt{-a}}\right), \quad a < 0, c > 0.$$

$$58. \int \frac{dx}{(ax^2 + c)^n} = \frac{1}{2(n-1)c} \frac{x}{(ax^2 + c)^{n-1}} + \frac{2n-3}{2(n-1)c} \int \frac{dx}{(ax^2 + c)^{n-1}}, \quad n \text{ a positive integer.}$$

$$59. \int x(ax^2 + c)^n dx = \frac{1}{2a} \frac{(ax^2 + c)^{n+1}}{n+1}, \quad n \neq -1.$$

$$60. \int \frac{x}{ax^2 + c} dx = \frac{1}{2a} \log(ax^2 + c).$$

$$61. \int \frac{dx}{x(ax^2 + c)} = \frac{1}{2c} \log\left(\frac{ax^2}{ax^2 + c}\right).$$

$$62. \int \frac{dx}{x^2(ax^2 + c)} = -\frac{1}{cx} - \frac{a}{c} \int \frac{dx}{ax^2 + c}.$$

$$63. \int \frac{x^2 dx}{ax^2 + c} = \frac{x}{a} - \frac{c}{a} \int \frac{dx}{ax^2 + c}.$$

$$64. \int \frac{x^n dx}{ax^2 + c} = \frac{x^{n-1}}{a(n-1)} - \frac{c}{a} \int \frac{x^{n-2} dx}{ax^2 + c}, \quad n \neq 1.$$

$$65. \int \frac{x^2 dx}{(ax^2 + c)^n} = -\frac{1}{2(n-1)a} \frac{x}{(ax^2 + c)^{n-1}} + \frac{1}{2(n-1)a} \int \frac{dx}{(ax^2 + c)^{n-1}}.$$

66. $\int \frac{dx}{x^2(ax^2 + c)^n} = \frac{1}{c} \int \frac{dx}{x^2(ax^2 + c)^{n-1}} - \frac{a}{c} \int \frac{dx}{(ax^2 + c)^n}.$
67. $\int \sqrt{x^2 \pm p^2} dx = \frac{1}{2} [x\sqrt{x^2 \pm p^2} \pm p^2 \log(x + \sqrt{x^2 \pm p^2})].$
68. $\int \sqrt{p^2 - x^2} dx = \frac{1}{2} \left[x\sqrt{p^2 - x^2} + p^2 \sin^{-1}\left(\frac{x}{p}\right) \right].$
69. $\int \frac{dx}{\sqrt{x^2 \pm p^2}} = \log(x + \sqrt{x^2 \pm p^2}).$
70. $\int \frac{dx}{\sqrt{p^2 - x^2}} = \sin^{-1}\left(\frac{x}{p}\right) \quad \text{or} \quad -\cos^{-1}\left(\frac{x}{p}\right).$
71. $\int \sqrt{ax^2 + c} dx = \frac{x}{2} \sqrt{ax^2 + c} + \frac{c}{2\sqrt{a}} \log(x\sqrt{a} + \sqrt{ax^2 + c}), \quad a > 0.$
72. $\int \sqrt{ax^2 + c} dx = \frac{x}{2} \sqrt{ax^2 + c} + \frac{c}{2\sqrt{-a}} \sin^{-1}\left(x\sqrt{\frac{-a}{c}}\right), \quad a < 0.$
73. $\int \frac{dx}{\sqrt{ax^2 + c}} = \frac{1}{\sqrt{a}} \log(x\sqrt{a} + \sqrt{ax^2 + c}), \quad a > 0.$
74. $\int \frac{dx}{\sqrt{ax^2 + c}} = \frac{1}{\sqrt{-a}} \sin^{-1}\left(x\sqrt{\frac{-a}{c}}\right), \quad a < 0.$
75. $\int x\sqrt{ax^2 + c} dx = \frac{1}{3a} (ax^2 + c)^{3/2}.$
76. $\int x^2\sqrt{ax^2 + c} dx = \frac{x}{4a} \sqrt{(ax^2 + c)^3} - \frac{cx}{8a} \sqrt{ax^2 + c} - \frac{c^2}{8\sqrt{a^3}} \log(x\sqrt{a} + \sqrt{ax^2 + c}), \quad a > 0.$
77. $\int x^2\sqrt{ax^2 + c} dx = \frac{x}{4a} \sqrt{(ax^2 + c)^3} - \frac{cx}{8a} \sqrt{ax^2 + c} - \frac{c^2}{8a\sqrt{-a}} \sin^{-1}\left(x\sqrt{\frac{-a}{c}}\right), \quad a < 0.$
78. $\int \frac{x dx}{\sqrt{ax^2 + c}} = \frac{1}{a} \sqrt{ax^2 + c}.$
79. $\int \frac{x^2 dx}{\sqrt{ax^2 + c}} = \frac{x}{a} \sqrt{ax^2 + c} - \frac{1}{a} \int \sqrt{ax^2 + c} dx.$
80. $\int \frac{\sqrt{ax^2 + c}}{x} dx = \sqrt{ax^2 + c} + \sqrt{c} \log\left(\frac{\sqrt{ax^2 + c} - \sqrt{c}}{x}\right), \quad c > 0.$
81. $\int \frac{\sqrt{ax^2 + c}}{x} dx = \sqrt{ax^2 + c} - \sqrt{-c} \tan^{-1} \frac{\sqrt{ax^2 + c}}{\sqrt{-c}}, \quad c < 0.$
82. $\int \frac{dx}{x\sqrt{p^2 \pm x^2}} = -\frac{1}{p} \log\left(\frac{p + \sqrt{p^2 \pm x^2}}{x}\right).$

83. $\int \frac{dx}{x\sqrt{x^2 - p^2}} = \frac{1}{p} \cos^{-1}\left(\frac{p}{x}\right) \text{ or } -\frac{1}{p} \sin^{-1}\left(\frac{p}{x}\right).$
84. $\int \frac{dx}{x\sqrt{ax^2 + c}} = \frac{1}{\sqrt{c}} \log\left(\frac{\sqrt{ax^2 + c} - \sqrt{c}}{x}\right), \quad c > 0.$
85. $\int \frac{dx}{x\sqrt{ax^2 + c}} = \frac{1}{\sqrt{-c}} \sec^{-1}\left(x\sqrt{\frac{-a}{c}}\right), \quad c < 0.$
86. $\int \frac{dx}{x^2\sqrt{ax^2 + c}} = -\frac{\sqrt{ax^2 + c}}{cx}.$
87. $\int \frac{x^n dx}{\sqrt{ax^2 + c}} = \frac{x^{n-1}\sqrt{ax^2 + c}}{na} - \frac{(n-1)c}{na} \int \frac{x^{n-2} dx}{\sqrt{ax^2 + c}}, \quad n > 0.$
88. $\int x^n \sqrt{ax^2 + c} dx = \frac{x^{n-1}(ax^2 + c)^{3/2}}{(n+2)a} - \frac{(n-1)c}{(n+2)a} \int x^{n-2} \sqrt{ax^2 + c} dx, \quad n > 0.$
89. $\int \frac{\sqrt{ax^2 + c}}{x^n} dx = -\frac{(ax^2 + c)^{3/2}}{c(n-1)x^{n-1}} - \frac{(n-4)a}{(n-1)c} \int \frac{\sqrt{ax^2 + c}}{x^{n-2}} dx, \quad n > 1.$
90. $\int \frac{dx}{x^n \sqrt{ax^2 + c}} = -\frac{\sqrt{ax^2 + c}}{c(n-1)x^{n-1}} - \frac{(n-2)a}{(n-1)c} \int \frac{dx}{x^{n-2} \sqrt{ax^2 + c}}, \quad n > 1.$
91. $\int (ax^2 + c)^{3/2} dx = \frac{x}{8}(2ax^2 + 5c)\sqrt{ax^2 + c} + \frac{3c^2}{8\sqrt{a}} \log(x\sqrt{a} + \sqrt{ax^2 + c}), \quad a > 0.$
92. $\int (ax^2 + c)^{3/2} dx = \frac{x}{8}(2ax^2 + 5c)\sqrt{ax^2 + c} + \frac{3c^2}{8\sqrt{-a}} \sin^{-1}\left(x\sqrt{\frac{-a}{c}}\right), \quad a < 0.$
93. $\int \frac{dx}{(ax^2 + c)^{3/2}} = \frac{x}{c\sqrt{ax^2 + c}}.$
94. $\int x(ax^2 + c)^{3/2} dx = \frac{1}{5a}(ax^2 + c)^{3/2}.$
95. $\int x^2(ax^2 + c)^{3/2} dx = \frac{x^3}{6}(ax^2 + c)^{3/2} + \frac{c}{2} \int x^2 \sqrt{ax^2 + c} dx.$
96. $\int x^n(ax^2 + c)^{3/2} dx = \frac{x^{n+1}(ax^2 + c)^{3/2}}{n+4} + \frac{3c}{n+4} \int x^n \sqrt{ax^2 + c} dx.$

97. $\int \frac{x \, dx}{(ax^2 + c)^{3/2}} = -\frac{1}{a\sqrt{ax^2 + c}}.$
98. $\int \frac{x^2 \, dx}{(ax^2 + c)^{3/2}} = -\frac{x}{a\sqrt{ax^2 + c}} + \frac{1}{a\sqrt{a}} \log(x\sqrt{a} + \sqrt{ax^2 + c}), \quad a > 0.$
99. $\int \frac{x^2 \, dx}{(ax^2 + c)^{3/2}} = -\frac{x}{a\sqrt{ax^2 + c}} + \frac{1}{a\sqrt{-a}} \sin^{-1}\left(x\sqrt{\frac{-a}{c}}\right), \quad a < 0.$
100. $\int \frac{x^3 \, dx}{(ax^2 + c)^{3/2}} = -\frac{x^2}{a\sqrt{ax^2 + c}} + \frac{2}{a^2}\sqrt{ax^2 + c}.$
101. $\int \frac{dx}{x(ax^n + c)} = \frac{1}{cn} \log\left(\frac{x^n}{ax^n + c}\right).$
102. $\int \frac{dx}{(ax^n + c)^m} = \frac{1}{c} \int \frac{dx}{(ax^n + c)^{m-1}} - \frac{a}{c} \int \frac{x^n \, dx}{(ax^n + c)^m}.$
103. $\int \frac{dx}{x\sqrt{ax^n + c}} = \frac{1}{n\sqrt{c}} \log\left(\frac{\sqrt{ax^n + c} - \sqrt{c}}{\sqrt{ax^n + c} + \sqrt{c}}\right), \quad c > 0.$
104. $\int \frac{dx}{x\sqrt{ax^n + c}} = \frac{2}{n\sqrt{-c}} \sec^{-1}\sqrt{\frac{-ax^n}{c}}, \quad c < 0.$
105. $\int x^{m-1}(ax^n + c)^p \, dx$
 $= \frac{1}{m + np} [x^m(ax^n + c)^p + npc \int x^{m-1}(ax^n + c)^{p-1} \, dx]$
 $= \frac{1}{cn(p+1)} [-x^m(ax^n + c)^{p+1} + (nt + np + n) \int x^{m-1}(ax^n + c)^{p+1} \, dx].$
 $= \frac{1}{a(m + np)} [x^{m-n}(ax^n + c)^{p+1} - (m-n)c \int x^{m-n-1}(ax^n + c)^p \, dx].$
 $= \frac{1}{mc} [x^m(ax^n + c)^{p+1} - (m + np + n) \int x^{m+n-1}(ax^n + c)^p \, dx].$
106. $\int \frac{x^m \, dx}{(ax^n + c)^p} = \frac{1}{a} \int \frac{x^{m-n} \, dx}{(ax^n + c)^{p-1}} - \frac{c}{a} \int \frac{x^{m-n} \, dx}{(ax^n + c)^p}.$
107. $\int \frac{dx}{x^m(ax^n + c)^p} = \frac{1}{c} \int \frac{dx}{x^m(ax^n + c)^{p-1}} - \frac{a}{c} \int \frac{dx}{x^{m-n}(ax^n + c)^p}.$

Expressions Containing $ax^2 + bx + c$

108. $\int \frac{dx}{ax^2 + bx + c} = \frac{1}{\sqrt{b^2 - 4ac}} \log\left(\frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}}\right), \quad b^2 > 4ac.$

$$109. \int \frac{dx}{ax^2 + bx + c} = \frac{2}{\sqrt{4ac - b^2}} \tan^{-1} \frac{2ax + b}{\sqrt{4ac - b^2}}, \quad b^2 < 4ac.$$

$$110. \int \frac{dx}{ax^2 + bx + c} = -\frac{2}{2ax + b}, \quad b^2 = 4ac.$$

$$111. \int \frac{dx}{(ax^2 + bx + c)^{n+1}} = \frac{2ax + b}{n(4ac - b^2)(ax^2 + bx + c)^n} \\ + \frac{2(2n - 1)a}{n(4ac - b^2)} \int \frac{dx}{(ax^2 + bx + c)^n}.$$

$$112. \int \frac{x dx}{ax^2 + bx + c} = \frac{1}{2a} \log(ax^2 + bx + c) - \frac{b}{2a} \int \frac{dx}{ax^2 + bx + c}.$$

$$113. \int \frac{x^2 dx}{ax^2 + bx + c} = \frac{x}{a} - \frac{b}{2a^2} \log(ax^2 + bx + c) \\ + \frac{b^2 - 2ac}{2a^2} \int \frac{dx}{ax^2 + bx + c}.$$

$$114. \int \frac{x^n dx}{ax^2 + bx + c} = \frac{x^{n-1}}{(n-1)a} - \frac{c}{a} \int \frac{x^{n-2} dx}{ax^2 + bx + c} - \frac{b}{a} \int \frac{x^{n-1} dx}{ax^2 + bx + c}.$$

$$115. \int \frac{xdx}{(ax^2 + bx + c)^{n+1}} = \frac{-(2c + bx)}{n(4ac - b^2)(ax^2 + bx + c)^n} \\ - \frac{b(2n - 1)}{n(4ac - b^2)} \int \frac{dx}{(ax^2 + bx + c)^n}.$$

$$116. \int \frac{x^m dx}{(ax^2 + bx + c)^{n+1}} = -\frac{x^{m-1}}{a(2n - m + 1)(ax^2 + bx + c)^n} \\ - \frac{n - m + 1}{2n - m + 1} \frac{b}{a} \int \frac{x^{m-1} dx}{(ax^2 + bx + c)^{n+1}} \\ + \frac{m - 1}{2n - m + 1} \frac{c}{a} \int \frac{x^{m-2} dx}{(ax^2 + bx + c)^{n+1}}$$

$$117. \int \frac{dx}{x(ax^2 + bx + c)} = \frac{1}{2c} \log\left(\frac{x^2}{ax^2 + bx + c}\right) - \frac{b}{2c} \int \frac{dx}{(ax^2 + bx + c)}.$$

$$118. \int \frac{dx}{x^2(ax^2 + bx + c)} = \frac{b}{2c^2} \log\left(\frac{ax^2 + bx + c}{x^2}\right) - \frac{1}{cx} \\ + \left(\frac{b^2}{2c^2} - \frac{a}{c}\right) \int \frac{dx}{(ax^2 + bx + c)}.$$

$$119. \int \frac{dx}{x^m(ax^2 + bx + c)^{n+1}} = -\frac{1}{(m-1)cx^{m-1}(ax^2 + bx + c)^n} \\ - \frac{(n+m-1)}{m-1} \frac{b}{c} \int \frac{dx}{x^{m-1}(ax^2 + bx + c)^{n+1}} \\ - \frac{(2n+m-1)}{m-1} \frac{a}{c} \int \frac{dx}{x^{m-2}(ax^2 + bx + c)^{n+1}}.$$

120.
$$\int \frac{dx}{x(ax^2 + bx + c)^n} = \frac{1}{2c(n-1)(ax^2 + bx + c)^{n-1}} - \frac{b}{2c} \int \frac{dx}{(ax^2 + bx + c)^n} + \frac{1}{c} \int \frac{dx}{x(ax^2 + bx + c)^{n-1}}.$$
121.
$$\int \frac{dx}{ax^2 + bx + c} = \frac{1}{\sqrt{a}} \log(2ax + b + 2\sqrt{a}\sqrt{ax^2 + bx + c}), \quad a > 0.$$
122.
$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{-a}} \sin^{-1} \frac{-2ax - b}{\sqrt{b^2 - 4ac}}, \quad a < 0.$$
123.
$$\int \frac{x dx}{\sqrt{ax^2 + bx + c}} = \frac{\sqrt{ax^2 + bx + c}}{a} - \frac{b}{2a} \int \frac{dx}{\sqrt{ax^2 + bx + c}}.$$
124.
$$\int \frac{x^n dx}{\sqrt{ax^2 + bx + c}} = \frac{x^{n-1}}{an} \sqrt{ax^2 + bx + c} - \frac{b(2n-1)}{2an} \int \frac{x^{n-1} dx}{\sqrt{ax^2 + bx + c}} - \frac{c(n-1)}{an} \int \frac{x^{n-2} dx}{\sqrt{ax^2 + bx + c}}.$$
125.
$$\int \sqrt{ax^2 + bx + c} dx = \frac{2ax + b}{4a} \sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8a} \int \frac{dx}{\sqrt{ax^2 + bx + c}}.$$
126.
$$\int x\sqrt{ax^2 + bx + c} dx = \frac{(ax^2 + bx + c)^{3/2}}{3a} - \frac{b}{2a} \int \sqrt{ax^2 + bx + c} dx.$$
127.
$$\int x^2\sqrt{ax^2 + bx + c} dx = \left(x - \frac{5b}{6a}\right) \frac{(ax^2 + bx + c)^{3/2}}{4a} + \frac{(5b^2 - 4ac)}{16a^2} \int \sqrt{ax^2 + bx + c} dx.$$
128.
$$\int \frac{dx}{x\sqrt{ax^2 + bx + c}} = -\frac{1}{\sqrt{c}} \log\left(\frac{\sqrt{ax^2 + bx + c} + \sqrt{c}}{x} + \frac{b}{2\sqrt{c}}\right), \quad c > 0.$$
129.
$$\int \frac{dx}{x\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{-c}} \sin^{-1} \frac{bx + 2c}{x\sqrt{b^2 - 4ac}}, \quad c < 0.$$
130.
$$\int \frac{dx}{x\sqrt{ax^2 + bx}} = -\frac{2}{bx} \sqrt{ax^2 + bx}, \quad c = 0.$$
131.
$$\int \frac{dx}{x^n\sqrt{ax^2 + bx + c}} = -\frac{\sqrt{ax^2 + bx + c}}{c(n-1)x^{n-1}} + \frac{b(3-2n)}{2c(n-1)} \int \frac{dx}{x^{n-1}\sqrt{ax^2 + bx + c}} + \frac{a(2-n)}{c(n-1)} \int \frac{dx}{x^{n-2}\sqrt{ax^2 + bx + c}}.$$
132.
$$\int \frac{dx}{(ax^2 + bx + c)^{3/2}} = -\frac{2(2ax + b)}{(b^2 - 4ac)\sqrt{ax^2 + bx + c}}, \quad b^2 \neq 4ac.$$

$$133. \int \frac{dx}{(ax^2 + bx + c)^{3/2}} = -\frac{1}{2\sqrt{a^3}(x + b/2a)^2}, \quad b^2 = 4ac.$$

Miscellaneous Algebraic Expression

$$134. \int \sqrt{2px - x^2} dx = \frac{1}{2}[(x - p)\sqrt{2px - x^2} + p^2 \sin^{-1}\left(\frac{x - p}{p}\right)].$$

$$135. \int \frac{dx}{\sqrt{2px - x^2}} = \cos^{-1}\left(\frac{p - x}{p}\right).$$

$$136. \int \frac{dx}{\sqrt{ax + b}\sqrt{cx + d}} = \frac{2}{\sqrt{-ac}} \tan^{-1} \sqrt{\frac{-c(ax + b)}{a(cx + d)}} \quad \text{or} \quad \frac{2}{\sqrt{ac}} \tanh^{-1} \sqrt{\frac{c(ax + b)}{a(cx + d)}}.$$

$$137. \int \sqrt{ax + b}\sqrt{cx + d} dx = \frac{(2acx + bc + ad)\sqrt{ax + b}\sqrt{cx + d}}{4ac} + \frac{(ad - bc)^2}{8ac} \int \frac{dx}{\sqrt{ax + b}\sqrt{cx + d}}.$$

$$138. \int \sqrt{\frac{cx + d}{ax + b}} dx = \frac{\sqrt{ax + b}\sqrt{cx + d}}{a} + \frac{(ad - bc)}{2a} \int \frac{dx}{\sqrt{ax + b}\sqrt{cx + d}}.$$

$$139. \int \sqrt{\frac{x + b}{x + d}} dx = \sqrt{x + d}\sqrt{x + b} + (b - d) \log(\sqrt{x + d} + \sqrt{x + b}).$$

$$140. \int \sqrt{\frac{1 + x}{1 - x}} dx = \sin^{-1}x - \sqrt{1 - x^2}.$$

$$141. \int \sqrt{\frac{p - x}{q + x}} dx = \sqrt{p - x}\sqrt{q + x} + (p + q) \sin^{-1} \sqrt{\frac{x + q}{p + q}}.$$

$$142. \int \sqrt{\frac{p + x}{q - x}} dx = -\sqrt{p + x}\sqrt{q - x} + (p + q) \sin^{-1} \sqrt{\frac{q - x}{p + q}}.$$

$$143. \int \frac{dx}{\sqrt{x - p}\sqrt{q - x}} = 2 \sin^{-1} \sqrt{\frac{x - p}{q - p}}.$$

Expressions Containing $\sin ax$

$$144. \int \sin u du = -\cos u, \quad \text{where } u \text{ is any function of } x.$$

$$145. \int \sin ax dx = -\frac{1}{a} \cos ax.$$

$$146. \int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a}.$$

147. $\int \sin^3 ax \, dx = -\frac{1}{a} \cos ax + \frac{1}{3a} \cos^3 ax.$
148. $\int \sin^4 ax \, dx = \frac{3x}{8} - \frac{3 \sin 2ax}{16a} - \frac{\sin^3 ax \cos ax}{4a}.$
149. $\int \sin^n ax \, dx = -\frac{\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n} \int \sin^{n-2} ax \, dx \quad (n \text{ pos. integer}).$
150. $\int \frac{dx}{\sin ax} = \frac{1}{a} \log\left(\tan \frac{ax}{2}\right) = \frac{1}{a} \log(\csc ax - \cot ax).$
151. $\int \frac{dx}{\sin^2 ax} = \int \csc^2 ax \, dx = -\frac{1}{a} \cot ax.$
152. $\int \frac{dx}{\sin^n ax} = -\frac{1}{a(n-1)} \frac{\cos ax}{\sin^{n-1} ax} + \frac{n-2}{n-1} \int \frac{dx}{\sin^{n-2} ax}, \quad n \text{ integer} > 1.$
153. $\int \frac{dx}{1 \pm \sin ax} = \mp \frac{1}{a} \tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right).$
154. $\int \frac{dx}{b + c \sin ax} = -\frac{2}{a\sqrt{b^2 - c^2}} \tan^{-1}\left[\sqrt{\frac{b-c}{b+c}} \tan\left(\frac{\pi}{4} - \frac{ax}{2}\right)\right], \quad b^2 > c^2.$
155. $\int \frac{dx}{b + c \sin ax} =$

$$\frac{-1}{a\sqrt{c^2 - b^2}} \log\left(\frac{c + b \sin ax + \sqrt{c^2 - b^2} \cos ax}{b + c \sin ax}\right), \quad c^2 > b^2.$$
156. $\int \sin ax \sin bx \, dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}, \quad a^2 \neq b^2.$
157. $\int 1 + \sin x \, dx = \pm 2\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right); \quad \text{use } + \text{ sign when}$
 $(8k-1)\frac{\pi}{2} < x \leq (8k+3)\frac{\pi}{2}, \text{ otherwise } -, \quad k \text{ an integer.}$
158. $\int \sqrt{1 - \sin x} \, dx = \pm 2\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right); \quad \text{use } + \text{ sign when}$
 $(8k-3)\frac{\pi}{2} < x \leq (8k+1)\frac{\pi}{2}, \text{ otherwise } -, \quad k \text{ an integer.}$

Expressions Involving $\cos ax$

159. $\int \cos u \, du = \sin u, \quad \text{where } u \text{ is any function of } x.$
160. $\int \cos ax \, dx = \frac{1}{a} \sin ax.$
161. $\int \cos^2 ax \, dx = \frac{x}{2} + \frac{\sin 2ax}{4a}.$

162. $\int \cos^3 ax \, dx = \frac{1}{a} \sin ax - \frac{1}{3a} \sin^3 ax.$
163. $\int \cos^4 ax \, dx = \frac{3x}{8} + \frac{3 \sin 2ax}{16a} + \frac{\cos^3 ax \sin ax}{4a}.$
164. $\int \cos^n ax \, dx = \frac{\cos^{n-1} ax \sin ax}{na} + \frac{n-1}{n} \int \cos^{n-2} ax \, dx \quad (n \text{ pos. integer}).$
165. $\int \frac{dx}{\cos ax} = \frac{1}{a} \log\left(\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)\right) = \frac{1}{a} \log(\tan ax + \sec ax).$
166. $\int \frac{dx}{\cos^2 ax} = \frac{1}{a} \tan ax.$
167. $\int \frac{dx}{\cos^n ax} = \frac{1}{a(n-1)} \frac{\sin ax}{\cos^{n-1} ax} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} ax}, \quad n \text{ integer} > 1.$
168. $\int \frac{dx}{1 + \cos ax} = \frac{1}{a} \tan \frac{ax}{2}.$
169. $\int \frac{dx}{1 - \cos ax} = -\frac{1}{a} \operatorname{ctn} \frac{ax}{2}.$
170. $\int \sqrt{1 + \cos x} \, dx = \pm \sqrt{2} \int \cos \frac{x}{2} \, dx = \pm 2\sqrt{2} \sin \frac{x}{2};$ use + when
 $(4k-1)\pi < x \leq (4k+1)\pi$, otherwise -, k an integer.
171. $\int \sqrt{1 - \cos x} \, dx = \pm \sqrt{2} \int \sin \frac{x}{2} \, dx = \mp 2\sqrt{2} \cos \frac{x}{2};$ use top signs
 when $4k\pi < x \leq (4k+2)\pi$, otherwise bottom signs.
172. $\int \frac{dx}{b + c \cos ax} = \frac{1}{a\sqrt{b^2 - c^2}} \tan^{-1}\left(\frac{\sqrt{b^2 - c^2} \sin ax}{c + b \cos ax}\right), \quad b^2 > c^2.$
173. $\int \frac{dx}{b + c \cos ax} = \frac{1}{a\sqrt{c^2 - b^2}} \tanh^{-1}\left(\frac{\sqrt{c^2 - b^2} \sin ax}{c + b \cos ax}\right), \quad c^2 > b^2.$
174. $\int \cos ax \cos bx \, dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}, \quad a^2 \neq b^2.$

Expressions Containing $\sin ax$ and $\cos ax$

175. $\int \sin ax \cos bx \, dx = -\frac{1}{2} \left(\frac{\cos(a-b)x}{a-b} + \frac{\cos(a+b)x}{a+b} \right), \quad a^2 \neq b^2.$
176. $\int \sin^n ax \cos ax \, dx = \frac{1}{a(n+1)} \sin^{n+1} ax, \quad n \neq -1.$
177. $\int \cos^n ax \sin ax \, dx = -\frac{1}{a(n+1)} \cos^{n+1} ax, \quad n \neq -1.$

$$178. \int \frac{\sin ax}{\cos ax} dx = -\frac{1}{a} \log(\cos ax).$$

$$179. \int \frac{\cos ax}{\sin ax} dx = \frac{1}{a} \log(\sin ax).$$

$$180. \int (b + c \sin ax)^n \cos ax dx = \frac{1}{ac(n+1)} (b + c \sin ax)^{n+1}, \quad n \neq -1.$$

$$181. \int (b + c \cos ax)^n \sin ax dx = -\frac{1}{ac(n+1)} (b + c \cos ax)^{n+1}, \quad n \neq -1.$$

$$182. \int \frac{\cos ax dx}{b + c \sin ax} = \frac{1}{ac} \ln|b + c \sin ax|.$$

$$183. \int \frac{\sin ax}{b + c \cos ax} dx = -\frac{1}{ac} \log(b + c \cos ax).$$

$$184. \int \frac{dx}{b \sin ax + c \cos ax} = \frac{1}{a\sqrt{b^2 + c^2}} \left[\log \left(\tan \frac{1}{2} \left(ax + \tan^{-1} \frac{c}{b} \right) \right) \right].$$

$$185. \int \frac{dx}{b + c \cos ax + d \sin ax} = \frac{-1}{a\sqrt{b^2 - c^2 - d^2}} \sin^{-1} U,$$

$$U \equiv \frac{c^2 + d^2 + b(c \cos ax + a \sin ax)}{\sqrt{c^2 + d^2}(b + c \cos ax + d \sin ax)} \quad \text{or} \quad \frac{1}{a\sqrt{c^2 + d^2 - b^2}} \log(V),$$

$$V \equiv \frac{c^2 + d^2 + b(c \cos ax + d \sin ax) + \sqrt{c^2 + d^2 - b^2}(c \sin ax - d \cos ax)}{\sqrt{c^2 + d^2}(b + c \cos ax + d \sin ax)},$$

$$b^2 \neq c^2 + d^2, \quad -\pi < ax < \pi.$$

$$186. \int \frac{dx}{b + c \cos ax + d \sin ax} = \frac{1}{ab} \left(\frac{b - (c + d) \cos ax + (c - d) \sin ax}{b + (c - d) \cos ax + (c + d) \sin ax} \right), \quad b^2 = c^2 + d^2.$$

$$187. \int \frac{\sin^2 ax dx}{b + c \cos^2 ax} = \frac{1}{ac} \sqrt{\frac{b+c}{b}} \tan^{-1} \left(\sqrt{\frac{b}{b+c}} \tan ax \right) - \frac{x}{c}.$$

$$188. \int \frac{\sin ax \cos ax dx}{b \cos^2 ax + c \sin^2 ax} = \frac{1}{2a(c-b)} \log(b \cos^2 ax + c \sin^2 ax).$$

$$189. \int \frac{dx}{b^2 \cos^2 ax - c^2 \sin^2 ax} = \frac{1}{2abc} \log \left(\frac{b \cos ax + c \sin ax}{b \cos ax - c \sin ax} \right).$$

$$190. \int \frac{dx}{b^2 \cos^2 ax - c^2 \sin^2 ax} = \frac{1}{abc} \tan^{-1} \left(\frac{c \tan ax}{b} \right).$$

$$191. \int \sin^2 ax \cos^2 ax dx = \frac{x}{8} - \frac{\sin 4ax}{32a}.$$

$$192. \int \frac{dx}{\sin ax \cos ax} = \frac{1}{a} \log(\tan ax).$$

$$193. \int \frac{dx}{\sin^2 ax \cos^2 ax} = \frac{1}{a} (\tan ax - \operatorname{ctn} ax).$$

194. $\int \frac{\sin^2 ax}{\cos ax} dx = \frac{1}{a} \left[-\sin ax + \log \left(\tan \left(\frac{ax}{2} + \frac{\pi}{4} \right) \right) \right].$
195. $\int \frac{\cos^2 ax}{\sin ax} dx = \frac{1}{a} \left[\cos ax + \log \left(\tan \frac{ax}{2} \right) \right].$
196. $\int \sin^m ax \cos^n ax dx = -\frac{\sin^{m-1} ax \cos^{n+1} ax}{a(m+n)} + \frac{m-1}{m+n} \int \sin^{m-2} ax \cos^n ax dx, \quad m, n > 0.$
197. $\int \sin^m ax \cos^n ax dx = \frac{\sin^{m+1} ax \cos^{n-1} ax}{a(m+n)} + \frac{n-1}{m+n} \int \sin^m ax \cos^{n-2} ax dx, \quad m, n > 0.$
198. $\int \frac{\sin^m ax}{\cos^n ax} dx = \frac{\sin^{m+1} ax}{a(n-1) \cos^{n-1} ax} - \frac{m-n+2}{n-1} \int \frac{\sin^m ax}{\cos^{n-2} ax} dx, \quad m, n > 0, n \neq 1.$
199. $\int \frac{\cos^n ax}{\sin^m ax} dx = \frac{-\cos^{n+1} ax}{a(m-1) \sin^{m-1} ax} + \frac{m-n-2}{(m-1)} \int \frac{\cos^n ax}{\sin^{m-2} ax} dx, \quad m, n > 0, m \neq 1.$
200. $\int \frac{dx}{\sin^m ax \cos^n ax} = \frac{1}{a(n-1)} \frac{1}{\sin^{m-1} ax \cos^{n-1} ax} + \frac{m+n-2}{(n-1)} \int \frac{dx}{\sin^m ax \cos^{n-2} ax}.$
201. $\int \frac{dx}{\sin^m ax \cos^n ax} = -\frac{1}{a(m-1)} \frac{1}{\sin^{m-1} ax \cos^{n-1} ax} + \frac{m+n-2}{(m-1)} \int \frac{dx}{\sin^{m-2} ax \cos^n ax}.$
202. $\int \frac{\sin^{2n} ax}{\cos ax} dx = \int \frac{(1 - \cos^2 ax)^n}{\cos ax} dx.$ (Expand, divide, and use 205).
203. $\int \frac{\cos^{2n} ax}{\sin ax} dx = \int \frac{(1 - \sin^2 ax)^n}{\sin ax} dx.$ (Expand, divide, and use 190).
204. $\int \frac{\sin^{2n+1} ax}{\cos ax} dx = \int \frac{(1 - \cos^2 ax)^n}{\cos ax} \sin ax dx.$ (Expand, divide, and use 218).
205. $\int \frac{\cos^{2n+1} ax}{\sin ax} dx = \int \frac{(1 - \sin^2 ax)^n}{\sin ax} \cos ax dx.$ (Expand, divide, and use 217).

**Expressions Containing $\tan ax$ or $\text{ctn } ax$
 ($\tan ax = 1/\text{ctn } ax$)**

$$206. \int \tan u \, du = -\log(\cos u) \quad \text{or} \quad \log(\sec u), \quad \text{where } u \text{ is any function of } x.$$

$$207. \int \tan ax \, dx = -\frac{1}{a} \log(\cos ax).$$

$$208. \int \tan^2 ax \, dx = \frac{1}{a} \tan ax - x.$$

$$209. \int \tan^3 ax \, dx = \frac{1}{2a} \tan^2 ax + \frac{1}{a} \log(\cos ax).$$

$$210. \int \tan^n ax \, dx = \frac{1}{a(n-1)} \tan^{n-1} ax - \int \tan^{n-2} ax \, dx, \quad n \text{ integer } > 1.$$

$$211. \int \text{ctn } u \, du = \log(\sin u) \quad \text{or} \quad -\log(\csc u), \quad \text{where } u \text{ is any function of } x.$$

$$212. \int \text{ctn}^2 ax \, dx = \int \frac{dx}{\tan^2 ax} = -\frac{1}{a} \text{ctn } ax - x.$$

$$213. \int \text{ctn}^3 ax \, dx = -\frac{1}{2a} \text{ctn}^2 ax - \frac{1}{a} \log(\sin ax).$$

$$214. \int \text{ctn}^n ax \, dx = \int \frac{dx}{\tan^n ax} \\ -\frac{1}{a(n-1)} \text{ctn}^{n-1} ax - \int \text{ctn}^{n-2} ax \, dx, \quad n \text{ integer } > 1.$$

$$215. \int \frac{dx}{b + c \tan ax} = \int \frac{\text{ctn } ax \, dx}{b \text{ctn } ax + c} \\ = \frac{1}{b^2 + c^2} \left[bx + \frac{c}{a} \log(b \cos ax + c \sin ax) \right].$$

$$216. \int \frac{dx}{b + c \text{ctn } ax} = \int \frac{\tan ax \, dx}{b \tan ax + c} \\ = \frac{1}{b^2 + c^2} \left(bx - \frac{c}{a} \log(c \cos ax + b \sin ax) \right).$$

$$217. \int \frac{dx}{\sqrt{b + c \tan^2 ax}} = \frac{1}{a\sqrt{b-c}} \sin^{-1} \left(\sqrt{\frac{b-c}{b}} \sin ax \right), \quad b \text{ pos.}, \quad b^2 > c^2.$$

Expressions Containing $\sec ax = 1/\cos ax$ or $\csc ax = 1/\sin ax$

218. $\int \sec u \, du = \log(\sec u + \tan u) = \log\left[\tan\left(\frac{u}{2} + \frac{\pi}{4}\right)\right]$,
 where u is any function of x .

219. $\int \sec ax \, dx = \frac{1}{a} \log\left[\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)\right]$.

220. $\int \sec^2 ax \, dx = \frac{1}{a} \tan ax$.

221. $\int \sec^3 ax \, dx = \frac{1}{2a} \left[\tan ax \sec ax + \log\left(\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)\right) \right]$.

222. $\int \sec^n ax \, dx = \frac{1}{a(n-1)} \frac{\sin ax}{\cos^{n-1} ax}$
 $+ \frac{n-2}{n-1} \int \sec^{n-2} ax \, dx, \quad n \text{ integer } > 1$.

223. $\int \csc u \, du = \log(\csc u - \cot u) = \log\left(\tan \frac{u}{2}\right)$,
 where u is any function of x .

224. $\int \csc ax \, dx = \frac{1}{a} \log\left(\tan \frac{ax}{2}\right)$.

225. $\int \csc^2 ax \, dx = -\frac{1}{a} \cot ax$.

226. $\int \csc^3 ax \, dx = \frac{1}{2a} \left[-\cot ax \csc ax + \log\left(\tan \frac{ax}{2}\right) \right]$.

227. $\int \csc^n ax \, dx = -\frac{1}{a(n-1)} \frac{\cos ax}{\sin^{n-1} ax} + \frac{n-2}{n-1} \int \csc^{n-2} ax \, dx, \quad n \text{ integer } > 1$.

Expressions Containing $\tan ax$ and $\sec ax$ or $\cot ax$ and $\csc ax$

228. $\int \tan u \sec u \, du = \sec u$, where u is any function of x .

229. $\int \tan ax \sec ax \, dx = \frac{1}{a} \sec ax$.

230. $\int \tan^n ax \sec^2 ax \, dx = \frac{1}{a(n+1)} \tan^{n+1} ax, \quad n \neq -1$.

$$231. \int \tan ax \sec^n ax \, dx = \frac{1}{an} \sec^n ax, \quad n \neq 0.$$

$$232. \int \operatorname{ctn} u \csc u \, du = -\csc u, \quad \text{where } u \text{ is any function of } x.$$

$$233. \int \operatorname{ctn} ax \csc ax \, dx = -\frac{1}{a} \csc ax.$$

$$234. \int \operatorname{ctn}^n ax \csc^2 ax \, dx = -\frac{1}{a(n+1)} \operatorname{ctn}^{n+1} ax, \quad n \neq -1.$$

$$235. \int \operatorname{ctn} ax \csc^n ax \, dx = -\frac{1}{an} \csc^n ax, \quad n \neq 0.$$

$$236. \int \frac{\csc^2 ax \, dx}{\operatorname{ctn} ax} = -\frac{1}{a} \log(\operatorname{ctn} ax).$$

Expressions Containing Algebraic and Trigonometric Functions

$$237. \int x \sin ax \, dx = \frac{1}{a^2} \sin ax - \frac{1}{a} x \cos ax.$$

$$238. \int x^2 \sin ax \, dx = \frac{2x}{a^2} \sin ax + \frac{2}{a^3} \cos ax - \frac{x^2}{a} \cos ax.$$

$$239. \int x^3 \sin ax \, dx = \frac{3x^2}{a^2} \sin ax - \frac{6}{a^4} \sin ax - \frac{x^3}{a} \cos ax + \frac{6x}{a^3} \cos ax.$$

$$240. \int x \sin^2 ax \, dx = \frac{x^2}{4} - \frac{x \sin 2ax}{4a} - \frac{\cos 2ax}{8a^2}.$$

$$241. \int x^2 \sin^2 ax \, dx = \frac{x^3}{6} - \left(\frac{x^2}{4a} - \frac{1}{8a^3} \right) \sin 2ax - \frac{x \cos 2ax}{4a^2}.$$

$$242. \int x^3 \sin^2 ax \, dx = \frac{x^4}{8} - \left(\frac{x^3}{4a} - \frac{3x}{8a^3} \right) \sin 2ax - \left(\frac{3x^2}{8a^2} - \frac{3}{16a^4} \right) \cos 2ax.$$

$$243. \int x \sin^3 ax \, dx = \frac{x \cos 3ax}{12a} - \frac{\sin 3ax}{36a^2} - \frac{3x \cos ax}{4a} + \frac{3 \sin ax}{4a^2}.$$

$$244. \int x^n \sin ax \, dx = -\frac{1}{a} x^n \cos ax + \frac{n}{a} \int x^{n-1} \cos ax \, dx, \quad n > 0.$$

$$245. \int \frac{\sin ax \, dx}{x} = ax - \frac{(ax)^3}{3 \cdot 3!} + \frac{(ax)^5}{5 \cdot 5!} - \dots$$

$$246. \int \frac{\sin ax \, dx}{x^m} = \frac{-1}{(m-1)} \frac{\sin ax}{x^{m-1}} + \frac{a}{(m-1)} \int \frac{\cos ax \, dx}{x^{m-1}}.$$

$$247. \int x \cos ax \, dx = \frac{1}{a^2} \cos ax + \frac{1}{a} x \sin ax.$$

248. $\int x^2 \cos ax \, dx = \frac{2x}{a^2} \cos ax - \frac{2}{a^3} \sin ax + \frac{x^2}{a} \sin ax.$
249. $\int x^3 \cos ax \, dx = \frac{(3a^2x^2 - 6) \cos ax}{a^4} + \frac{(a^2x^3 - 6x) \sin ax}{a^3}.$
250. $\int x \cos^2 ax \, dx = \frac{x^2}{4} + \frac{x \sin 2ax}{4a} + \frac{\cos 2ax}{8a^2}.$
251. $\int x^2 \cos^2 ax \, dx = \frac{x^3}{6} + \left(\frac{x^2}{4a} - \frac{1}{8a^3} \right) \sin 2ax + \frac{x \cos 2ax}{4a^2}.$
252. $\int x^3 \cos^2 ax \, dx = \frac{x^4}{8} + \left(\frac{x^3}{4a} - \frac{3x}{8a^3} \right) \sin 2ax + \left(\frac{3x^2}{8a^2} - \frac{3}{16a^4} \right) \cos 2ax.$
253. $\int x \cos^3 ax \, dx = \frac{x \sin 3ax}{12a} + \frac{\cos 3ax}{36a^2} + \frac{3x \sin ax}{4a} + \frac{3 \cos ax}{4a^2}.$
254. $\int x^n \cos ax \, dx = \frac{1}{a} x^n \sin ax - \frac{n}{a} \int x^{n-1} \sin ax \, dx, \quad n \text{ pos.}$
255. $\int \frac{\cos ax \, dx}{x} = \log(ax) - \frac{(ax)^2}{2 \cdot 2!} + \frac{(ax)^4}{4 \cdot 4!} - \dots.$
256. $\int \frac{\cos ax}{x^m} \, dx = -\frac{1}{(m-1)} \frac{\cos ax}{x^{m-1}} - \frac{a}{(m-1)} \int \frac{\sin ax \, dx}{x^{m-1}}.$

Expressions Containing Exponential and Logarithmic Functions

257. $\int e^u \, du = e^u,$ where u is any function of x .
258. $\int b^u \, du = \frac{b^u}{\log(b)},$ where u is any function of x .
259. $\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad \int b^{ax} \, dx = \frac{b^{ax}}{a \log(b)}.$
260. $\int x e^{ax} \, dx = \frac{e^{ax}}{a^2} (ax - 1), \quad \int x b^{ax} \, dx = \frac{x b^{ax}}{a \log(b)} - \frac{b^{ax}}{a^2 (\log(b))^2}.$
261. $\int x^2 e^{ax} \, dx = \frac{e^{ax}}{a^3} (a^2 x^2 - 2ax + 2).$
262. $\int x^n e^{ax} \, dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx, \quad n \text{ pos.}$
263. $\int x^n e^{ax} \, dx = \frac{e^{ax}}{a^{n+1}} [(ax)^n - n(ax)^{n-1} + n(n-1)(ax)^{n-2} - \dots + (-1)^n n!], \quad n \text{ pos. integ.}$

$$264. \int x^n e^{-ax} dx = -\frac{e^{-ax}}{a^{n+1}} [(ax)^n + n(ax)^{n-1} + n(n-1)(ax)^{n-2} + \cdots + n!], \quad n \text{ pos. integ.}$$

$$265. \int x^n b^{ax} dx = \frac{x^n b^{ax}}{a \log(b)} - \frac{n}{a \log(b)} \int x^{n-1} b^{ax} dx, \quad n \text{ pos.}$$

$$266. \int \frac{e^{ax}}{x} dx = \log(x) + ax + \frac{(ax)^2}{2 \cdot 2!} + \frac{(ax)^3}{3 \cdot 3!} + \cdots.$$

$$267. \int \frac{e^{ax}}{x^n} dx = \frac{1}{n-1} \left(-\frac{e^{ax}}{x^{n-1}} + a \int \frac{e^{ax}}{x^{n-1}} dx \right), \quad n \text{ integ. } > 1.$$

$$268. \int \frac{dx}{b + ce^{ax}} = \frac{1}{ab} [ax - \log(b + ce^{ax})].$$

$$269. \int \frac{e^{ax} dx}{b + ce^{ax}} = \frac{1}{ac} \log(b + ce^{ax}).$$

$$270. \int \frac{dx}{be^{ax} + ce^{-ax}} = \frac{1}{a\sqrt{bc}} \tan^{-1} \left(e^{ax} \sqrt{\frac{b}{c}} \right), \quad b \text{ and } c \text{ pos.}$$

$$271. \int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx).$$

$$272. \int e^{ax} \sin bx \sin cx dx = \frac{e^{ax} [(b-c) \sin(b-c)x + a \cos(b-c)x]}{2[a^2 + (b-c)^2]} - \frac{e^{ax} [(b+c) \sin(b+c)x + a \cos(b+c)x]}{2[a^2 + (b+c)^2]}.$$

$$273. \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx).$$

$$274. \int e^{ax} \cos bx \cos cx dx = \frac{e^{ax} [(b-c) \sin(b-c)x + a \cos(b-c)x]}{2[a^2 + (b-c)^2]} + \frac{e^{ax} [(b+c) \sin(b+c)x + a \cos(b+c)x]}{2[a^2 + (b+c)^2]}.$$

$$275. \int e^{ax} \sin bx \cos cx dx = \frac{e^{ax} [a \sin(b-c)x - (b-c) \cos(b-c)x]}{2[a^2 + (b-c)^2]} + \frac{e^{ax} [a \sin(b+c)x - (b+c) \cos(b+c)x]}{2[a^2 + (b+c)^2]}.$$

$$276. \int e^{ax} \sin bx \sin(bx+c) dx = \frac{e^{ax} \cos c}{2a} - \frac{e^{ax} [a \cos(2bx+c) + 2b \sin(2bx+c)]}{2(a^2 + 4b^2)}.$$

$$277. \int e^{ax} \cos bx \cos(bx+c) dx = \frac{e^{ax} \cos c}{2a} + \frac{e^{ax} [a \cos(2bx+c) + 2b \sin(2bx+c)]}{2(a^2 + 4b^2)}.$$

$$278. \int e^{ax} \sin bx \cos(bx + c) dx = -\frac{e^{ax} \sin c}{2a} + \frac{e^{ax}[a \sin(2bx + c) - 2b \cos(2bx + c)]}{2(a^2 + 4b^2)}.$$

$$279. \int e^{ax} \cos bx \sin(bx + c) dx = \frac{e^{ax} \sin c}{2a} + \frac{e^{ax}[a \sin(2bx + c) - 2b \cos(2bx + c)]}{2(a^2 + 4b^2)}.$$

$$280. \int xe^{ax} \sin bx dx = \frac{xe^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) - \frac{e^{ax}}{(a^2 + b^2)^2}[(a^2 - b^2) \sin bx - 2ab \cos bx].$$

$$281. \int xe^{ax} \cos bx dx = \frac{xe^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) - \frac{e^{ax}}{(a^2 + b^2)^2}[(a^2 - b^2) \cos bx + 2ab \sin bx].$$

$$282. \int e^{ax} \cos^n bx dx = \frac{e^{ax} (\cos^{n-1} bx)(a \cos bx + nb \sin bx)}{a^2 + n^2 b^2} + \frac{n(n-1)b^2}{a^2 + n^2 b^2} \int e^{ax} \cos^{n-2} bx dx.$$

$$283. \int e^{ax} \sin^n bx dx = \frac{e^{ax} (\sin^{n-1} bx)(a \sin bx - nb \cos bx)}{a^2 + n^2 b^2} + \frac{n(n-1)b^2}{a^2 + n^2 b^2} \int e^{ax} \sin^{n-2} bx dx.$$

$$284. \int \log(ax) dx = x \log(ax) - x.$$

$$285. \int x \log(ax) dx = \frac{x^2}{2} \log(ax) - \frac{x^2}{4}.$$

$$286. \int x^2 \log(ax) dx = \frac{x^3}{3} \log(ax) - \frac{x^3}{9}.$$

$$287. \int [\log(ax)]^2 dx = x[\log(ax)]^2 - 2x \log(ax) + 2x.$$

$$288. \int [\log(ax)]^n dx = x[\log(ax)]^n - n \int [\log(ax)]^{n-1} dx, \quad n \text{ pos.}$$

$$289. \int x^n \log(ax) dx = x^{n+1} \left(\frac{\log(ax)}{n+1} - \frac{1}{(n+1)^2} \right), \quad n \neq -1.$$

$$290. \int x^n [\log(ax)]^m dx = \frac{x^{n+1}}{n+1} [\log(ax)]^m - \frac{m}{n+1} \int x^n [\log(ax)]^{m-1} dx.$$

$$291. \int \frac{[\log(ax)]^n}{x} dx = \frac{[\log(ax)]^{n+1}}{n+1}, \quad n \neq -1.$$

$$292. \int \frac{dx}{x \log(ax)} = \log[\log(ax)].$$

$$293. \int \frac{dx}{x(\log(ax))^n} = -\frac{1}{(n-1)(\log(ax))^{n-1}}.$$

$$294. \int \frac{x^n dx}{(\log(ax))^m} = \frac{-x^{n+1}}{(m-1)(\log(ax))^{m-1}} + \frac{n+1}{m-1} \int \frac{x^n dx}{(\log(ax))^{m-1}}, \quad m \neq 1.$$

$$295. \int \frac{x^n dx}{\log(ax)} = \frac{1}{a^{n+1}} \int \frac{e^y dy}{y}, \quad y = (n+1) \ln|ax|.$$

$$296. \int \frac{x^n dx}{\log(ax)} = \frac{1}{a^{n+1}} [\log|\log(ax)| + (n+1) \log(ax) + \frac{(n+1)^2 [\log(ax)]^2}{2 \cdot 2!} + \frac{(n+1)^3 [\log(ax)]^3}{3 \cdot 3!} + \dots].$$

$$297. \int \frac{dx}{\log(ax)} = \frac{1}{a} [\log[\log(ax)] + \log(ax) + \frac{(\log(ax))^2}{2 \cdot 2!} + \frac{(\log(ax))^3}{3 \cdot 3!} + \dots].$$

$$298. \int \sin[\log(ax)] dx = \frac{x}{2} [\sin[\log(ax)] - \cos[\log(ax)]].$$

$$299. \int \cos[\log(ax)] dx = \frac{x}{2} [\sin[\log(ax)] + \cos[\log(ax)]].$$

$$300. \int e^{ax} \log(bx) dx = \frac{1}{a} e^{ax} \log(bx) - \frac{1}{a} \int \frac{e^{ax}}{x} dx.$$

Expressions Containing Inverse Trigonometric Functions

$$301. \int \sin^{-1} ax dx = x \sin^{-1} ax + \frac{1}{a} \sqrt{1 - a^2 x^2}.$$

$$302. \int (\sin^{-1} ax)^2 dx = x(\sin^{-1} ax)^2 - 2x + \frac{2}{a} \sqrt{1 - a^2 x^2} \sin^{-1} ax.$$

$$303. \int x \sin^{-1} ax dx = \frac{x^2}{2} \sin^{-1} ax - \frac{1}{4a^2} \sin^{-1} ax + \frac{x}{4a} \sqrt{1 - a^2 x^2}.$$

$$304. \int x^n \sin^{-1} ax dx = \frac{x^{n+1}}{n+1} \sin^{-1} ax - \frac{a}{n+1} \int \frac{x^{n+1} dx}{\sqrt{1 - a^2 x^2}}, \quad n \neq -1.$$

305.
$$\int \frac{\sin^{-1} ax \, dx}{x} = ax + \frac{1}{2 \cdot 3 \cdot 3} (ax)^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 5} (ax)^5$$

$$+ \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7 \cdot 7} (ax)^7 + \dots, \quad a^2 x^2 < 1.$$
306.
$$\int \frac{\sin^{-1} ax \, dx}{x^2} = -\frac{1}{x} \sin^{-1} ax - a \log \left(\frac{1 + \sqrt{1 - a^2 x^2}}{ax} \right).$$
307.
$$\int \cos^{-1} ax \, dx = x \cos^{-1} ax - \frac{1}{a} \sqrt{1 - a^2 x^2}.$$
308.
$$\int (\cos^{-1} ax)^2 \, dx = x(\cos^{-1} ax)^2 - 2x - \frac{2}{a} \sqrt{1 - a^2 x^2} \cos^{-1} ax.$$
309.
$$\int x \cos^{-1} ax \, dx = \frac{x^2}{2} \cos^{-1} ax - \frac{1}{4a^2} \cos^{-1} ax - \frac{x}{4a} \sqrt{1 - a^2 x^2}.$$
310.
$$\int x^n \cos^{-1} ax \, dx = \frac{x^{n+1}}{n+1} \cos^{-1} ax + \frac{a}{n+1} \int \frac{x^{n+1} \, dx}{\sqrt{1 - a^2 x^2}}, \quad n \neq -1.$$
311.
$$\int \frac{\cos^{-1} ax \, dx}{x} = \frac{\pi}{2} \ln|ax| - ax - \frac{1}{2 \cdot 3 \cdot 3} (ax)^3$$

$$- \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 5} (ax)^5 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7 \cdot 7} (ax)^7 - \dots, \quad a^2 x^2 < 1.$$
312.
$$\int \frac{\cos^{-1} ax \, dx}{x^2} = -\frac{1}{x} \cos^{-1} ax + a \log \left(\frac{1 + \sqrt{1 - a^2 x^2}}{ax} \right).$$
313.
$$\int \tan^{-1} ax \, dx = x \tan^{-1} ax - \frac{1}{2a} \log(1 + a^2 x^2).$$
314.
$$\int x^n \tan^{-1} ax \, dx = \frac{x^{n+1}}{n+1} \tan^{-1} ax - \frac{a}{n+1} \int \frac{x^{n+1} \, dx}{1 + a^2 x^2}, \quad n \neq -1.$$
315.
$$\int \frac{\tan^{-1} ax \, dx}{x^2} = -\frac{1}{x} \tan^{-1} ax - \frac{a}{2} \log \left(\frac{1 + a^2 x^2}{a^2 x^2} \right).$$
316.
$$\int \operatorname{ctn}^{-1} ax \, dx = x \operatorname{ctn}^{-1} ax + \frac{1}{2a} \log(1 + a^2 x^2).$$
317.
$$\int x^n \operatorname{ctn}^{-1} ax \, dx = \frac{x^{n+1}}{n+1} \operatorname{ctn}^{-1} ax + \frac{a}{n+1} \int \frac{x^{n+1} \, dx}{1 + a^2 x^2}, \quad n \neq -1.$$
318.
$$\int \frac{\operatorname{ctn}^{-1} ax \, dx}{x^2} = -\frac{1}{x} \operatorname{ctn}^{-1} ax + \frac{a}{2} \log \left(\frac{1 + a^2 x^2}{a^2 x^2} \right).$$
319.
$$\int \sec^{-1} ax \, dx = x \sec^{-1} ax - \frac{1}{a} \log(ax + \sqrt{a^2 x^2 - 1}).$$
320.
$$\int x^n \sec^{-1} ax \, dx = \frac{x^{n+1}}{n+1} \sec^{-1} ax \pm \frac{1}{n+1} \int \frac{x^n \, dx}{\sqrt{a^2 x^2 - 1}}, \quad n \neq -1;$$

use + sign when $\pi/2 < \sec^{-1} ax < \pi$; - sign when $0 < \sec^{-1} ax < \pi/2$.

$$321. \int \csc^{-1} ax \, dx = x \csc^{-1} ax + \frac{1}{a} \log(ax + \sqrt{a^2 x^2 - 1}).$$

$$322. \int x^n \csc^{-1} ax \, dx = \frac{x^{n+1}}{n+1} \csc^{-1} ax \pm \frac{1}{n+1} \int \frac{x^n \, dx}{\sqrt{a^2 x^2 - 1}}, \quad n \neq -1;$$

use + sign when $0 < \csc^{-1} ax < \pi/2$; - sign when $-\pi/2 < \csc^{-1} ax < 0$.

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